Based on National
Curriculum of Pakistan 2022-23

Model Textbook of

MATHEMATICS

Grade-9

National Book Foundation
as
Federal Textbook Board
Islamabad

Based on National Curriculum of Pakistan 2022-23

Model Textbook of

Mathematics

Science Group

Grade

09

National Curriculum Council
Ministry of Federal Education and Professional Training

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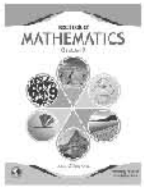
Model Textbook of **Mathematics**
for Grade 9



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Preface

This Model Textbook for Mathematics grade 9 has been developed by NBF according to the National Curriculum of Pakistan 2022-2023. The aim of this textbook is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building upon the foundation of learning from the previous grades. A key emphasis of the present textbook is on creating real life linkages of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they go up the learning curve and for them to fully grasp the conceptual basis that will be built upon in subsequent grades.

After amalgamation of the efforts of experts and experienced authors, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its books. The present book features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement and the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this book.

May Allah guide and help us (Ameen).

Dr. Raja Mazhar Hameed
Managing Director

بِسْمِ اللّٰہِ الرَّحْمٰنِ الرَّحِیْمِ

اللہ کے نام سے شروع جو بڑا مہربان، نہایت رحم والا ہے

Content

UNIT
01

REAL NUMBERS

In this unit the students will be able to:

- Recall the history of numbers.
- Recall the set of real numbers as a union of sets of rational and irrational numbers.
- Depict real numbers on the number line.
- Demonstrate a number with terminating and non-terminating recurring decimals on the number line.
- Give decimal representation of rational and irrational numbers.
- Know the properties of real numbers.
- Explain the concept of radicals and radicands.
- Diferentiate between radical form and exponential form of an expression.
- Transform an expression given in radical form to an exponential form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.

A number is an abstract idea used in *counting* and *measuring*. A symbol which represents a number is called a numeral, but in common usage the word number is used for both the idea and the symbol. In addition to their use in counting and measuring, numerals are often used for labels (telephone numbers), for ordering (serial numbers), and for codes (ISBN, i.e. International Standard Book Number). In mathematics, the definition of number has been extended over the years to include such numbers as zero, negative numbers, rational numbers, irrational numbers, real numbers and complex numbers.

1.1 History of Real Numbers

Can you imagine a world without numbers?

In our daily conversations, our domestic activities and at our jobs, we cannot spend a single day without using numbers. Our lives will be quite strange without involvement of numbers. In this way one can imagine about the life of a person in the era when numbers were not discovered.

There is an interesting story regarding how humans started using numbers for the first time. The story is about a shepherd boy who used pebbles to count his goats /sheep he sent for grazing each day to avoid missing any. The number discovered in that way were simply the counting numbers which are now called **Natural numbers**. People usually used their fingers or sticks to count objects and were symbolized by tally marks. They counted objects by carving tally marks into cave walls, bones, woods or stones about 30000 BC. In fact, the earliest numerals recorded so far were simple marks for small numbers and special symbol for 10. These symbols appeared in early Egyptians inscriptions around 3500 BC to 3000 BC.

Sumerians Contribution to Numeral System

Some historians believe that Sumerians were first to use symbols for numerals

Around 5000 BC. Sumerian was a Great civilization settled at the Fertile Crescent area near present Iraq. They had a great contribution to the development of number system and basic Mathematics as they are considered to be very advanced in many fields at that era. e.g.

- They were first to construct buildings.
- They initiated modern agriculture, so they were keen about fundamental calculations.
- They depended on rise and fall of sun to estimate time, which showed that they were keen observers of angles and geometry.
- They used cuneiforms on clay tablets.

Interesting Information

The first record of existence of tally marks is on a leg bone of a baboon dating prior to 30000 BC. The bone has 29 clear notches in a row. It was discovered in South Africa.

Babylonians Contribution to Numeral System

Babylonians system of numerals based on Earlier Sumerian numeral system basically, as they were descendants of Sumerian. They relied upon a series of cuneiform marks to represent digits. This was a sexagesimal system of numbers (system with base 60). This concept is still in use today as in division of time we use 60 minutes, 60 seconds etc.

Although they carried complicated algebraic calculations and knew about the concept of nothingness but they didn't symbolize it ever rather they used a space between digits to represent zero. It made their numbers and calculations quite ambiguous. Around Babylon, clay was abundant so they made a lot of use of clay tablets impression with cuneiforms. The cuneiforms and numerals occur together in some documents from about 3000 BC. They seem to have some convention regarding the use. Cuneiform was always used for wages due while wages paid were written in curvilinear.

Greek Contribution to Numeral System

The early Greeks, like their predecessors Egyptians and Babylonians, also repeated units to 9 and probably had " – " symbols for ten and " O " for 100. The Greek system of abbreviation in numbers called Attick numerals is present in the record of 5th century BC but probably was used much earlier. Like Babylonians and Romans, the ancient Greeks knew about nothingness but did not symbolize the concept.

Researchers had discovered that many other civilizations developed positional notation of numbers independently including the Ancient Chinese and Aztecs.

Romans Contribution to Numeral System

Romans used tally marks on tally sticks or tally bones. Like early humans they also used V to represent five and X for ten. Ancient Romans incorporated these symbols into their seven symbol system. The Roman empire was very vast and this system of numerals was used thus widely throughout Europe from early 2000 years ago to late middle ages. Like the Babylonians, the Ancient Roman Numeral system lacked to symbolize nothingness. This system was maintained for nearly 2000 years in commerce and scientific literature.

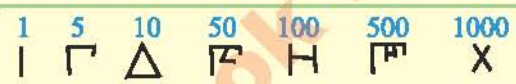
i.

Egyptian hieroglyphic numerals, 3300 B.C.

ii.

Babylonian cuneiform numerals, 2000 B.C.

iii.



Early Greek numerals, 5 B.C.

iv.

Roman numerals, 100 A.D.

v.

Mayan numerals, 300 A.D.

vi.

0 1 2 3 4 5 6 7 8 9

Hindu-Arabic digits, present day

Discovery of Zero (the sooper Hero)

Historians believed that Mayans living in central America used the idea of zero in their calendar system but they were isolated from other world so it couldnot caoe in outer world. Some historians give the crown of symbolizing nothingness to the Indian mathematician and astronomer Brahmagupta in 628 AD. It is also narrated that a great Muslim Mathematician Abu Muhammad Musa al Khwarizmi, who was also an astronomer and a geographer, contributed much to our modern understanding of maths. He described a number system based on 10 numerals from zero to nine in 7^{th} century in his book 'The use of the Hindu Numerals'(كتاب فی استعمال الاعداد الهندی) . He called this new digit as 'Sifr'. This useful system was soon adopted by Arabs.

Zero in the Europe

Fibonacci, the son of an Italian merchant discovered that Arab traders were using accounting system based on 0-9 numerals. He quickly understood its importance and improved book keeping and accounting in Europe. In 1202, he published a book describing this number system. He elaborated in the book about practical application i.e. how to convert one currency into other, calculation of profit and losses and other important business needs. In Italy 'Sifr' became 'zefero' which later became zero.

Discovery of zero brought a new set of numbers called set of whole numbers and it reduced the hurdles in calculation and understanding numbers.

Negative Numbers

The Chinese Mathematician Diophantus was most probably the first who used negative numbers around 200 BC in his work. He represented the amount of debt or loss. Then in 7^{th} century the Indian Brahmagupta wrote rules for adding, subtracting, multiplying and dividing negative numbers. The discovery of negative numbers gave existence to the set of Integers.

Rational Numbers

Pythagoras the Greek mathematician used fractions for the first time in mathematics around 500BC, which was infact the discovery of rational numbers. The word rational came from ratio.

Irrational Numbers

Soon after the discovery of rational numbers by Pythagoras, one of his early follower, Hippasus of Metapontum was working to find the hypotenuse of a right isosceles triangle with 2 equal sides of length 1 unit. He came with a strange answer ($\sqrt{2}$) and concluded that the answer is not reasonable number because it cannot be written in a/b form. He called such numbers as irrational (meaning stupid, nonsense or not reasonable). It is narrated that Hippasus was drowned in the sea for his new concept of numbers as it was against their self made religion.

Real numbers

The set of real numbers was thus completed with the discovery of rational and irrational numbers. We know today that the set of reals contain both and there is no such real number which is neither rational nor irrational.

1.2 Introduction to Real Numbers

Earlier mathematicians particularly Richard Dedekind have precisely defined the concept of real numbers which include both rational numbers such as $\frac{2}{9}$ and irrational numbers such as $\sqrt{2}$.

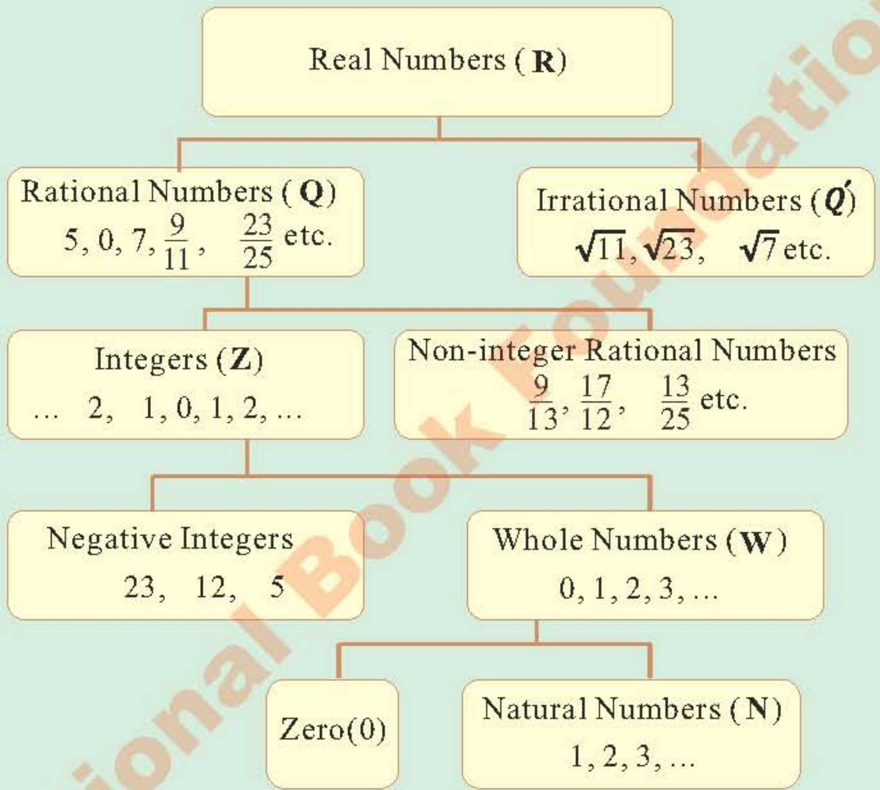
The real numbers are used to solve real life problems such as finding velocity, speed or distance, the profit or loss of a business, the difference in stock market etc.

Q Set of Rational Numbers	Q' Set of Irrational Numbers

$$R = Q \cup Q'$$
$$Q \cap Q' = \phi.$$

Key Fact

Set of rational numbers (Q) and set of irrational numbers (Q') are disjoint i.e., $Q \cap Q' = \phi$. But $R = Q \cup Q'$, then Q and Q' are called exhaustive sets.



1.2.1 Number Line

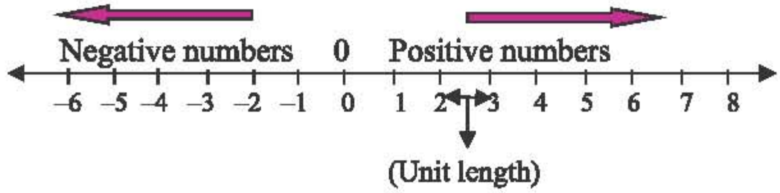
We use a number line to visualize real numbers and their relation to each other. To construct a number line, we choose a point corresponding to the number '0'. Points at equally spaced intervals are then associated with the integers. The positive integers are to the right of '0' and negative integers are to the left of 0. All other real numbers are associated with a point which is called the ***coordinate* of that point**. The point associated with zero is called the *origin*.

Enlighten Yourself

Number line is also called a real line because we express real numbers on it.

Usually few integers are shown on a number line to indicate the unit length of the line, that is, the distance between any two consecutive integers.

A number line is shown in figure below.



1.3 Properties of Real Numbers

The basic properties of real numbers are w.r.t addition and multiplication. In this section, some of the properties of these operations are reviewed. The following results are true for any real numbers *a*, *b* and *c*.

Name of the property	With respect to		Examples	
	$+$	$\times$	$+$	$\times$
Closure	$a+b\in R$	$a\bullet b\in R$	$6+4=10\in R$	$6\times 4=24\in R$
Commutative	$a+b=b+a$	$a\bullet b=b\bullet a$	$4+7=7+4$ $=11$	$4\times 7=7\times 4$ $=28$
Associative	$a+(b+c)=(a+b)+c$	$a\bullet(b\bullet c)=(a\bullet b)\bullet c$	$4+(6+8)=(4+6)+8$ $=18$	$4\times(6\times 8)=(4\times 6)\times 8$ $=192$
Identity	$a+0=a=0+a$	$a\bullet 1=1\bullet a=a$	$6+0=0+6$ $=6$	$6\times 1=1\times 6$ $=6$
Inverse	$a+(-a)=-a+a$ $=0$	$a\bullet\frac{1}{a}=\frac{1}{a}\bullet a=1$ $a\neq 0$	$14+(-14)=-14+14$ $=0$	$14\times\frac{1}{14}=\frac{1}{14}\times 14$ $=1$

Key Fact

i. 0 is the additive identity and 1 is the multiplicative identity of real numbers.

ii. $-a$ is the additive inverse of a and $\frac{1}{a}$ $(a\neq 0)$ is the multiplicative inverse of a.

1.3.1 Distributive Property of Multiplication over Addition

The distributive property involves both operations i.e., addition and multiplication.
Distributive property says, for all real numbers a, b, and c:

$$a(b+c)=ab+ac$$

Example 1: If $a=5$, $b=6$ and $c=9$, then verify that
$a(b+c)=ab+ac$.

Solution: $a(b+c)=5(6+9)=5(15)=75$
$ab+ac = 5(6)+5(9)=30+45=75$

Thus:

$$a(b+c)=ab+ac$$

Thinking Corner

- Which number has the additive inverse the number itself.
- Do all real numbers have their multiplicative inverses.
- Which number has no multiplicative inverses.

Example 2: Use the distributive property to simplify

$$32\left(\frac{1}{8}+\frac{1}{4}\right).$$

Solution: $32\left(\frac{1}{8}+\frac{1}{4}\right)=32\times\frac{1}{8}+32\times\frac{1}{4}$ $\boxed{a\,(b+c)=ab+ac}$

$$=4+8=12$$

> **Enlighten Yourself**
> Distributive property of multiplication over subtraction,
> $a\,(b-c)=ab-ac$ also holds.

1.3.1 Properties of Equality and Inequality of Real Numbers

Equality and Inequality Symbols

There are three symbols which can be used to show the possible relations between any two real numbers 'a' and 'b'. These are <, = and >, where '=' is equality symbol and '< and >' are inequality symbols.

Following table shows the use of these symbols.

Read	Write
a is less than b	$a<b$
a is equal to b	$a=b$
a is greater than b	$a>b$

> **Historya Mystery**
> The symbol < is used for "is less than" and > is used for "is greater than". These were introduced by Thomas & Harriot around 1630.

If a and b are real numbers, then only one of the following statement is true

(i) $a<b$ (ii) $a=b$ (iii) $a>b$

This property is known as **Trichotomy Property.**

> A mathematical statement with the equality sign is called an '**equality**'. A mathematical statement in which we do not use the symbol of equality but other symbols like '<' or '>' or both, is called an '**inequality**'.

Properties of Equality of Real Numbers

The following properties are true for any real numbers a, b and c.

Name of property	General statement
Reflexive	$a=a$
Symmetric	If $a=b$ then $b=a$
Transitive	If $a=b$ and $b=c$ then $a=c$
Additive	If $a=b$ then $a+c=b+c$

Multiplicative	If $a = b$ then $ac = bc$ or $ca = cb$, where $c \neq 0$
Cancellation w.r.t. addition	If $a + c = b + c$ then $a = b$
Cancellation w.r.t. multiplication	If $ac = bc$ then $a = b$ If $ca = cb$ then $a = b$ } where $c \neq 0$

Properties of Inequality of Real Numbers

The following properties are true for any real numbers *a*, *b* and *c*.

Name of property	General statement	Examples:
Trichotomy	Either $a > b$ or $a = b$ or $a < b$	If $2 < 3$, then $2 \neq 3$ and $2 \not> 3$
Transitive	If $a < b$ and $b < c$ then $a < c$ If $a > b$ and $b > c$ then $a > c$	If $3 < 5$ and $5 < 7$, then $3 < 7$ If $-2 > -5$ and $-5 > -7$, then $-2 > -7$
Additive	If $a < b$, then $a + c < b + c$ If $a > b$, then $a + c > b + c$	If $3 < 5$, then $3 + 10 < 5 + 10$ If $-5 > -8$, then $-5 + 2 > -8 + 2$
Multiplicative	(a) For $c < 0$ and $a, b \in R$, (i) If $a > b$ then $ac < bc$ (ii) If $a < b$, then $ac > bc$	(a) For $-4 < 0$, (i) If $3 > 2$, then $3(-4) < 2(-4)$ $-12 < -8$ (ii) If $3 < 5$, then $3(-4) > 5(-4)$ $-12 > -20$
Multiplicative	(b) For $c > 0$ and $a, b \in R$, (i) If $a > b$, then $ac > bc$ (ii) If $a < b$, then $ac < bc$ (iii) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$	(b) For $3 > 0$ (i) If $5 > 2$, then $5(3) > 2(3)$ $15 > 6$ (ii) If $2 < 3$, then $2(5) < 3(5)$ $10 < 15$ (iii) If $2 < 3$, then $\frac{1}{2} > \frac{1}{3}$
Cancellation w.r.t addition	If $a + c < b + c$ then $a < b$ If $a + c > b + c$ then $a > b$	If $2 + 3 < 5 + 3$ then $2 < 5$ If $4 + 3 > 2 + 3$ then $4 > 2$
Cancellation w.r.t multiplication	If $ac < bc$ and $c > 0$ then $a < b$ if $ac < bc$ and $c < 0$ then $a > b$ if $ac > bc$ and $c > 0$ then $a > b$ if $ac > bc$ and $c < 0$ then $a < b$	If $2 \times 3 < 4 \times 3$ then $2 < 4$ If $-3 \times 2 > -3 \times 4$ then $2 < 4$

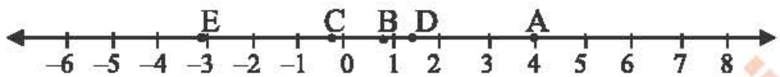
Key Fact

- The inequality sign remains unchanged if a positive number is multiplied to both the sides of an inequality.
- The inequalities are reversed if a negative number is multiplied to both the sides of an inequality.

Example 3: Show the following numbers on a number line.

(a) 4 (b) $\frac{7}{8}$ (c) $-\frac{1}{3}$ (d) $\sqrt{2}$ (e) $-\pi$

Solution:



(a) The number 4 is four units to the right of **0**, therefore, A is representing 4 on number line.

(b) $\frac{7}{8}$ = 0.875 is between 0 and 1, which is a terminating decimal. Point B in the figure is representing $\frac{7}{8}$ on the number line.

(c) $-\frac{1}{3} = -0.333\ldots$ or $-0.\bar{3}$ is between 0 and –1, which is a recurring decimal. Point C in the figure is representing $-\frac{1}{3}$ on the number line.

(d) Since $\sqrt{2} = 1.414213\ldots$, is between 1 and 2. Point D in the figure is the location of $\sqrt{2}$.

(e) Since $-\pi = -3.14159\ldots$ is between –3 and –4. Point E in the figure is the location of $-\pi$.

1.3.2 Representation of Real Numbers on Number Line

The representation of real numbers on a number line is called graphing the real numbers or graph of the real numbers.

Example 4: Represent the following sets of real numbers on a number line.

(a) $x \le 2$ (b) $-6 < x < 4$

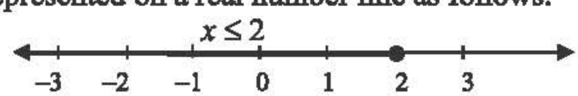
(c) $x > -4$ (d) $-2 \le x < 1$

Thinking zone

Try to imagine the numbers less than or equal to 2 and relate the words at least or at most which ever suitable in this case.

Solution:

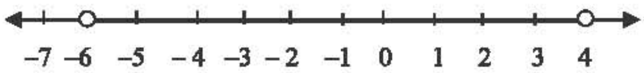
(a) The inequality $x \le 2$ specifies all real numbers less than or equal to 2. This set is represented on a real number line as follows.



A filled circle indicate that 2 is included in the set.

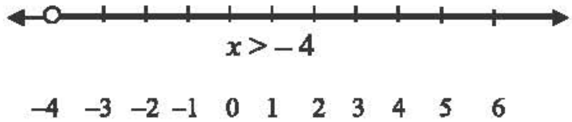
(b) The inequality $-6 < x < 4$ specifies all real numbers between – 6 and 4, as shown on the number line.

$-6 < x < 4$



We use hollow circle to indicate that both –6 and 4 are not included in the set.

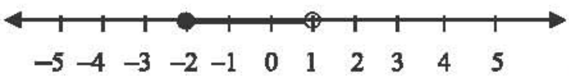
(c) $x > -4$ specifies all real numbers greater than -4.



We use hollow dot to indicate that – 4 is not included in the set.

(d) $-2 \le x < 1$ is the set of all real numbers between –2 and 1 including –2 but excluding 1.

$-2 < x < 1$



EXERCISE 1.1

1. Represent each number on the number line.

(i) $\frac{3}{4}$ (ii) $\frac{-1}{3}$ (iii) $4\frac{1}{2}$ (iv) $-\sqrt{8}$

(v) $\sqrt{8}$ (vi) $-4\frac{1}{2}$ (vii) $\frac{1}{3}$ (viii) $-\frac{7}{8}$

2. Identify the property that justifies.

(i) $1 \times (y - 2) = y - 2$ (ii) $(0.2)\,5 = 1$

(iii) $(x + 2) + y = y + (x + 2)$ (iv) $-(3b) + (3b) = 0$

(v) $(x + 5) - 1 = x + (5 - 1)$ (vi) $-3\,(2 - y) = -6 + 3y$

3. Represent the following on a number line.

(i) $x < 0$ (ii) $-3 < x < 3$ (iii) $x \ge -8$

(iv) $x > 0$ (v) $x < -3$ (vi) $-4 < x \le 4$

4. Identify the properties of equality and inequality of real numbers that justifies the statement.

(i) $9x = 9x$ (ii) If $x + 2 = y$ and $y = 2x - 3$, then $x + 2 = 2x - 3$

(iii) If $2x + 3 = y$, then $y = 2x + 3$ (iv) If $3 < 4$, then $-3 > -4$

(v) If $2y + 2w = p$ and $p = 50$, then $2y + 2w = 50$

(vi) If $x + 4 > y + 4$, then $x > y$ (vii) If $2 < 5$ and $5 < 9$, then $2 < 9$

(viii) If $-18 < -16$, then $9 > 8$

1.4 Radical and Radicands

1.4.1 Square Root

A square root of a positive number 'n' is another number 'm' whose square is 'n.' Any positive number has two square roots, which are additive inverses of each other.

For example, 4 is a square root of 16, because $(4)^2=16$ and -4 is also a square root of 16, because $(-4)^2=16$. Therefore, the two square roots of 16 are -4 and 4, which are additive inverse of each other.

1.4.2 Principal Square Root

Positive square roots are called 'principal square roots'. e.g. $\sqrt{25}=5, \sqrt{81}=9$ etc.

In expressions like $\sqrt{25}$ entire $\sqrt{25}$ is called a **square radical or radical**. The symbol $\sqrt{\ }$ is called a **radical sign** and the number '25' under the radical sign is called the **radicand**.

Definition of n^{th} Root

For any real numbers 'a' and 'b' and any positive integer $n > 1$, if $a^n = b$, then $a = b^{\frac{1}{n}}$, where 'n' is the index of the radical.

We read $a^n = b$ as 'b is the n^{th} power of a' and $a = b^{\frac{1}{n}}$ as 'a is the n^{th} root of b'.

For example, $y^3 = x$, then $y = x^{\frac{1}{3}} = \sqrt[3]{x}$

Here 3 is the index of radical and y is the cubic root of x.

Example 5:

Radical Form	Index of the Radical	Radicand
$\sqrt[3]{35}$	3	35
$\sqrt[5]{\frac{xy}{z}}$	5	$\frac{xy}{z}$
$\sqrt{-(xyz)^4}$	2	$-(xyz)^4$

Historya Mystery

The radical sign was first used in 1525 AD and was written as "$\surd$".

Key Fact

(i) $\sqrt{b} = \sqrt[2]{b}$ i.e. $\sqrt{\ }$ and $\sqrt[2]{\ }$ are equivalent.

(ii) There is no real number that is a square root of a negative number.

e.g. $\sqrt{-16} \neq 4$, since $(+4)^2 \neq -16$. In Mathematics 'imaginary numbers' are defined to handle the square root of negative numbers.

1.4.3 Properties of Radicals

1. Product and Quotient Rules for Radicals

For any integer n > 1 and for all real numbers 'a' and 'b' for which the operations are defined

(i) $\sqrt[n]{a}.\sqrt[n]{b}=\sqrt[n]{ab}$ ⟶ Product rule for radicals

e.g. $\sqrt[3]{8}\times\sqrt[3]{27}=\sqrt[3]{8\times 27}$

(ii) and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$ ⟶ Quotient rule for radicals e.g. $\dfrac{\sqrt[3]{64}}{\sqrt[3]{8}}=\sqrt[3]{\dfrac{64}{8}}$

Example 6: Use the product rule for radicals to simplify the following. Assume that all variables represent positive numbers.

(a) $\sqrt{2a}.\sqrt{7b}$ (b) $\sqrt[4]{\dfrac{1}{x}}.\sqrt[4]{\dfrac{2}{y}}$ (c) $\sqrt[3]{3}.\sqrt{2}$

Solution:

(a) $\sqrt{2a}.\sqrt{7b}=\sqrt{14ab}$

(b) $\sqrt[4]{\dfrac{1}{x}}.\sqrt[4]{\dfrac{2}{y}}=\sqrt[4]{\dfrac{2}{xy}}$

(c) The product rule for radicals does not apply to $\sqrt[3]{3}.\sqrt{2}$, because the indices are not same.

Key Fact

- The product and quotient rules for radicals apply only if the indices are same.
- The plural of index is indices.

2. Reducing the Index

If the index of the radical and the exponent of the radicand have a common factor, the expression can be written with a smaller index.

We will explain it with the help of an example. Consider $\sqrt[12]{9^6}$, here the index of the radical is 12 and exponent of radicand is 6. Also 6 is the common factor of 12 and 6 . So we can write the expression with a smaller index as follows.

$$\sqrt[12]{9^6}=9^{\frac{6}{12}}=9^{\frac{1}{2}}=\sqrt{9}=3$$

Example 7: Write the given expression with a smaller index. Assume that variable t represents positive numbers.

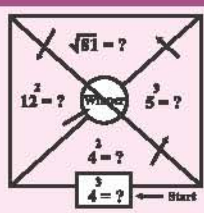
$\sqrt[15]{t^{10}}$

Solution : $\sqrt[15]{t^{10}}=t^{\frac{10}{15}}$

$=t^{\frac{2}{3}}=\sqrt[3]{t^2}$

Math Play Ground

1. Make a hopscotch on ground as give:
2. Ask a player to start hoping and doing sums in each box.
3. If reaches in circle without falling and doing correct calculations, he/she wins.



1.5 Laws of Exponents / Indices

1.5.1 Base and Exponents

We use exponent to indicate the repeated multiplication of the same factors.
The exponent indicates that how many times a factor, called the *base,* occurs in the multiplication form.

e.g.

$$3.3.3.3 = 3^4$$

Exponent (4), Base (3)

The expression 3^4 is an exponential form read as 'three to the fourth power', where as 3.3.3.3 is the factored form. The words 'square' and 'cube' are sometimes used for exponents '2' and '3' respectively, rather than 'to the second power' and 'to the third power'.

Historya Mystery

Rene Descartes (1596 – 1650) was the first mathematician who extensively used exponential notation as it is used today. However, for some unknown reason, he always used xx for x^2.

Key Fact

When the exponent is a natural number, the base can be any real number.
We use an exponent as a convenient way to write repeated multiplication.

1.5.2 Rational Exponents

When we perform operations with exponents, we have to define a zero exponent and a negative exponent. This may lead us to define a rational exponent.

Definition of $b^{\frac{1}{n}}$

If b is a real number and n is a positive integer, then $b^{\frac{1}{n}} = \sqrt[n]{b}$ e.g. $8^{\frac{1}{3}} = \sqrt[3]{8}$.

Definition of $b^{\frac{m}{n}}$

If m and n are positive integers with no common factor except 1 and $n \neq 0$, then
$b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m = (\sqrt[n]{b})^m$ for all real numbers b. e.g. $36^{\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^3$.

We can use this definition to write expressions with rational exponents as radicals.
The number 'n' indicates the index of the radical and the number 'm' indicates the power to which the radical is to be raised.

The procedure for evaluating $b^{\frac{m}{n}}$ can be summarized as follows.

I. Determine the nth root of b.
II. Raise the result to the m power.

Example 8: In the following table:

(a) perform the operation in column-A and compare the result to the value of radical in column- B.

Column-A (exponential form)	Column-B (radical form)
(i) $9^{\frac{1}{2}}$	$\sqrt{9}$
(ii) $64^{\frac{1}{3}}$	$\sqrt[3]{64}$
(iii) $81^{\frac{1}{4}}$	$\sqrt[4]{81}$
(iv) $32^{\frac{1}{5}}$	$\sqrt[5]{32}$

(b) what do you observe about the denominator of the exponent and the index of the radical?

Solution:

(a)

(i) $9^{\frac{1}{2}} = 3$ and $\sqrt{9} = 3$

(ii) $64^{\frac{1}{3}} = (4^3)^{\frac{1}{3}} = 4$ and $\sqrt[3]{64} = 4$

(iii) $81^{\frac{1}{4}} = 3$ and $\sqrt[4]{81} = (3^4)^{\frac{1}{4}} = 3$

(iv) $32^{\frac{1}{5}} = 2$ and $\sqrt[5]{32} = 2$

(b) In each part, the exponential and the radical expression have the same value. The denominator of the exponent is the same as the index of the radical.

Example 9: Simplify:

(a) $8^{\frac{2}{3}}$ (b) $36^{\frac{3}{2}}$

Solution:

(a) $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$

$= \left[\left(2^3\right)^{\frac{1}{3}}\right]^2$

$= (2)^2 = 4$

(b) $36^{\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^3$

$= \left[\left(6^2\right)^{\frac{1}{2}}\right]^3$

$= (6)^3 = 216$

Math Play Ground

Jump on the numbers which are squares of natural numbers to go out.

IN

81	35	47	63	40
49	36	25	28	32
74	15	4	100	9
20	13	6	89	1

OUT

Negative Rational Exponents

For integral exponents, we define:

$a^{-n} = \frac{1}{a^n}$ provided $a \neq 0$. e.g. $8^{-3} = \frac{1}{8^3}$.

We can extend this definition to negative rational exponents.

If m and n are any two integers such that one of them is negative and they have no common factor other than 1 and if $b \neq 0$, then $b^{\frac{-m}{n}} = \frac{1}{b^{\frac{m}{n}}}$ for all $b \in R$, for which $b^{\frac{m}{n}}$ is defined.

Example 10: Simplify.

(a) $16^{\frac{-3}{4}}$ (b) $\left(\frac{16}{25}\right)^{\frac{-1}{2}}$

Solution:

(a) $16^{\frac{-3}{4}} = \frac{1}{(16)^{\frac{3}{4}}}$

$= \frac{1}{(2^4)^{\frac{3}{4}}}$

$= \frac{1}{2^3}$

$= \frac{1}{8}$

(b) $\left(\frac{16}{25}\right)^{\frac{-1}{2}}$

$= \left(\frac{25}{16}\right)^{\frac{1}{2}}$

$= \sqrt{\frac{16}{25}}$

$= \sqrt{\frac{5^2}{4^2}} = \frac{5}{4}$

Example 11: Write exponential expressions as an equivalent radical expression.

(a) $(-7)^{\frac{2}{3}}$ (b) $(2)^{\frac{3}{5}}$

Solution: (a) $(-7)^{\frac{2}{3}} = (-7)^{2\times\frac{1}{3}}$

$=(49)^{\frac{1}{3}}$

$=\sqrt[3]{49}$

(b) $(2)^{\frac{3}{5}} = \left[(2)^3\right]^{\frac{1}{5}}$

$= \sqrt[5]{8}$

1.5.3 Properties of Exponents

If m and n are rational numbers, then for non zero real numbers a and b for which the expressions are defined, the following are the properties of exponents.

i) $a^m . a^n = a^{m+n} \rightarrow$ Product rule

ii) $\frac{a^m}{a^n} = a^{m-n} \rightarrow$ Quotient rule

iii) $a^{-n} = \frac{1}{a^n} \rightarrow$ Definition of negative exponent,

iv) $(a^m)^n = a^{mn} \rightarrow$ Power of a power rule

v) $(ab)^n = a^n b^n \rightarrow$ Power of a product rule

vi) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \rightarrow$ Power of a quotient rule

vii) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \rightarrow$ Negative power of a quotient rule

Example 12: Use the properties of exponents to evaluate each of the following.

(a) $(5^6)^{\frac{1}{2}}$ (b) $2^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$

Solution:

(a) $\left(5^{6}\right)^{\frac{1}{2}}$

$= 5^{6\times\frac{1}{2}} = 5^{3}$

$= 125$

(b) $2^{1/2} \cdot 8^{1/2}$

$= (2 \cdot 8)^{\frac{1}{2}} = (16)^{\frac{1}{2}}$

$= (4^{2})^{\frac{1}{2}} = 4$

Example 13: Use properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (Write all results with positive exponents.)

(a) $a^{\frac{1}{3}}(a^{\frac{5}{3}} - a^{\frac{-2}{3}})$

(b) $\left[\dfrac{x^{\frac{1}{2}}}{y^{3}}\right]^{-\frac{1}{3}}$

Solution: (a) $a^{\frac{1}{3}}(a^{\frac{5}{3}} - a^{\frac{-2}{3}})$

$= a^{\frac{1}{3}}a^{\frac{5}{3}} - a^{\frac{1}{3}}a^{\frac{-2}{3}}$

$= a^{\frac{1}{3}+\frac{5}{3}} - a^{\frac{1}{3}-\frac{2}{3}}$

$= a^{\frac{6}{3}} - a^{\frac{-1}{3}}$

$= a^{2} - \dfrac{1}{a^{\frac{1}{3}}} = \dfrac{a^{\frac{2}{3}} - 1}{a^{\frac{1}{3}}}$

(b) $\left[\dfrac{x^{\frac{1}{2}}}{y^{3}}\right]^{-\frac{1}{3}}$

$= \left[\dfrac{y^{3}}{x^{\frac{1}{2}}}\right]^{\frac{1}{3}} = \dfrac{\left(y^{3}\right)^{\frac{1}{3}}}{\left(x^{\frac{1}{2}}\right)^{\frac{1}{3}}}$

$= \dfrac{y}{x^{\frac{1}{6}}}$

EXERCISE 1.2

1. By using the property of product and quotient rule for radicals, write each expression as a single radical and simplify.

(i) $\sqrt[3]{6} \cdot \sqrt[3]{6}$ (ii) $\sqrt[5]{4} \,.\, \sqrt[5]{8}$ (iii) $\sqrt[4]{x} \,.\, \sqrt[4]{x^{3}}$

(iv) $\sqrt{10} \cdot \sqrt[3]{11}$ (v) $\dfrac{\sqrt[4]{x^{7}}}{\sqrt[4]{x^{5}}}$ (vi) $\dfrac{\sqrt[3]{5000}}{\sqrt[3]{5}}$

(vii) $\dfrac{\sqrt[2]{500}}{\sqrt[2]{5}}$ (viii) $\sqrt[2]{10} \cdot \sqrt[3]{7}$

2. Write each exponential expression as an equivalent radical expression and simplify if possible.

(i) $(216)^{\frac{2}{3}}$ (ii) $(29)^{\frac{1}{2}}$ (iii) $\left(\dfrac{1}{32}\right)^{\frac{1}{5}}$

(iv) $(216)^{-\frac{2}{3}}$ (v) $(1000)^{\frac{1}{3}}$ (vi) $\left(\dfrac{1}{39}\right)^{-\frac{1}{2}}$

3. Write each radical expression as an equivalent exponential expression and simplify if possible.

(i) $\left(\sqrt[3]{5}\right)^2$ (ii) $\left(\sqrt[4]{10}\right)^8$ (iii) $-\left(\sqrt[3]{6}\right)^6$

(iv) $\left(\sqrt[3]{6}\right)^6$ (v) $-\left(\sqrt[3]{5}\right)^2$ (vi) $-\left(\sqrt[4]{10}\right)^8$

4. Use the properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (write all results with positive exponents.)

(i) $\dfrac{16^{\frac{1}{5}}.16^{\frac{1}{4}}}{16^{\frac{-3}{10}}}$ (ii) $7^{\frac{-1}{3}}\left(7^{\frac{5}{3}} - 7^{\frac{4}{3}}\right)$ (iii) $\dfrac{2^{\frac{2}{3}}.2^{\frac{1}{7}}}{2^{\frac{1}{2}}}$

(iv) $\dfrac{3^{\frac{-1}{2}}.3^{\frac{1}{2}}}{3^{\frac{1}{2}}}$ (v) $\left(\dfrac{36^{\frac{1}{2}}}{8^{\frac{1}{2}}}.\dfrac{6^{\frac{1}{2}}}{27^{\frac{1}{2}}}\right)^3$ (vi) $\left(\dfrac{2187\ a^5 b^{17}}{a^{12} b^3}\right)^{\frac{1}{7}}$

(vii) $\sqrt[4]{\dfrac{a^3}{b^3}} \times \sqrt[4]{\dfrac{b^3}{c^3}} \times \sqrt[4]{\dfrac{c^3}{a^3}}$

5. Use suitable laws of exponents to show that

$$\left(\frac{x^p}{x^q}\right)^{p+q} \times \left(\frac{y^q}{y^r}\right)^{q+r} \times \left(\frac{z^r}{z^p}\right)^{r+p} \times x^{q^2} \times y^{r^2} \times z^{p^2} = x^{p^2} \times y^{q^2} \times z^{r^2}$$

1.5.4 Application of Real Numbers in Daily Life

All the numbers we use in our daily life situations are Real numbers. We cannot imagine life without numbers. For instance we use natural numbers in counting our objects in the pantry, books in the library, animals or birds at a farm, stock taking in inventory of a factory etc. Similarly, we have a vast use of integers while recording or understanding temperature, gain or loss, rise or fall etc. Rational numbers have also the vast contribution in daily life situations such as use of ratio, proportion, fractions and percentages in financial matters like income, expenditure, savings, and payment of wages to employees, rents of buildings, profit, loss sharing in business managements, risk calculations. The irrational numbers as obvious from the name are not reasonable or they don't make a sense for non-mathematicians. But for mathematicians they have really big scope of usage. Engineers, technicians, opticians while working with circles, spheres or cylinders and finding their areas, perimeters or volume, which include π are working with irrational numbers. Then we find irrational numbers like in architecture, navigation and fluid mechanics, where transcendental functions are in frequent use.

Example 14:

A cooking oil company produces four types of oils in packing of 1 litre , 5 litre & 10 litre. The inventory is shown in the table.

Name / packing size	1 litre	5 litre	10 litre
Cooking oil-I	5000	2500	1000
Cooking oil-II	5000	2500	1000
Cooking oil-III	5000	2500	1000
Cooking oil-IV	5000	2500	1000

After removal of 40% of an item, it is to be replenished. The daily removal of 1lire cooking oil-II packing is 20% and for 10 litre cooking oil-IV packing, daily removal is 5%. Find

a) Number of daily removed 1 litre cooking oil-II packing.
b) After how many days, 1 litre cooking oil-II are to be replenished?
c) Number of daily removed 10 litre cooking oil-IV packing.
d) After how many days, 10 litre cooking oil-IV packing are to be replenished?

Solution:

Total 1 litre cooking oil-II packing in inventory = 5000

a) Number of daily removed 1lire cooking oil-II packing = 20% of 5000

$= \frac{20}{100} \times 5000$

$= 1000$

b) After 40% removal, replenishment is to be made.

Here 40% of 5000 = 2000

After two days the replenishment is due.

c) Number of daily removed 10 lire cooking oil-IV packing = 5% of 1000

$= \frac{5}{100} \times 1000$

$= 50$

d) Here 40% of 1000 = 400 but packs removed per day are 50.

Therefore $400 \div 50 = 8$

After 8 days the replenishment is due.

EXERCISE 1.3

1. On his last bank statement, Qasim had a balance of Rs. 1,75,000 in his checking account. He wrote one cheque for Rs. 45,790 and another for Rs. 112,921. What is his current balance?
2. Last week Wajid drove 283.4 km on 16.2 litres of petrol. He says that he averaged about 1.75 km/liter. Is his answer reasonable? Explain.
3. Salma bought 3.2 yard of fabric for a total price of Rs. 139.2. How much did the fabric cost per yard?

4. Momina walks 3.5 km/h. She took a 12 h walk. How far did she walk.
5. The hiking club went on a 7day trip. Each day they hiked between 5.5 and 7.5 miles. It is reasonable to assume that clubbing the days the club hiked.
   a. Less than 35 miles
   b. Between 35 and 55 miles
   c. Equally 55 miles
   d. More than 55 miles
6. For a class party the students council purchased 42 balloons at Rs. 1.85 each. What is the total amount the student council paid for the balloons?
7. A group of friends made 4-yard long rectangular banner. They paid Rs. 3.75 per yard for the fabric and Rs.9 for the firm to go around the banner, 10-yard perimeter. What was the width of the banner?
8. A shoe factory has an asset for Rs. 2000,000 of which $\frac{3}{5}$ is the capital and rest is the debt. Find the amount of capital and debt. (Asset = capital + debt)
9. World lowest temperature in past 100 years was recorded to be -89.2 ℃ at Vostok, Antarctica on July 21, 1983. Covert this temperature into Fahrenheit and Kelvin scales.

   $$\left(F = \frac{9}{5}C + 32 \;, \qquad K = ℃ + 273\right)$$
10. A company was penalized by the government act for low quality production. If the company has 3 share holders. Farah, Maryam and Tehreem investing in the ratios of 1 : 2 : 3 and the amount of penalty is Rs. 456,868.97. Find the amount of penalty paid by each of 3 share holders.

KEY POINTS

- Real numbers are union of rational and irrational numbers.
- Basic properties of real numbers are
  - Closure
  - Commutative
  - Associative
  - Identity
  - Inverse
  - Distributive property
- Properties of equality of real numbers:
  - Reflexive
  - Symmetric
  - Transitive
  - Additive
  - Multiplicative
  - Cancellation
- Properties of inequality of real numbers:
  - Trichotomy property
  - Transitive
  - Additive
  - Multiplicative
  - Cancellation
- Positive square roots are called principal square roots.
- For any real numbers a and b and any positive integer n >1 if $a^n = b$, then a is the nth root of b, symbolically it is represented as $a = \sqrt[n]{b}$.

- The symbol $\sqrt{\ }$ is called the radical sign, the number n is called index of the radical and b is called radicand.
- Laws of exponents

(i) $a^m . a^n = a^{m+n} \rightarrow$ Product rule (ii) $\frac{a^m}{a^n} = a^{m-n} \rightarrow$ Quotient rule

(iii) $a^{-n} = \frac{1}{a^n} \rightarrow$ Negative exponent (iv) $(a^m)^n = a^{mn} \rightarrow$ Power to a power rule

(v) $(ab)^n = a^n b^n \rightarrow$ Power of a product rule

(vi) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \rightarrow$ Power of a quotient rule

MISCELLANEOUS EXERCISE 1

1. Encircle the correct option in the following.

(i) a (b + c – d) equals

(a) a (b + c + d) (b) ac + ab – ad (c) ab + ac + ad (d) ab – ac – ad

(ii) $a^r.a^{-s} \div a^s$ is

(a) a^{r-s} (b) a^{r+2s} (c) $a^r.a^{2s}$ (d) $\frac{a^r}{a^{2s}}$

(iii) $\sqrt[n]{ab}$ is equal to

(a) $\sqrt{ab}$ (b) $n(ab)$ (c) $(ab)^n$ (d) $(ab)^{\frac{1}{n}}$

(iv) Which number is self-multiplicative inverse?

(a) 3 (b) –3 (c) –1 (d) 0

(v) If $a > 0$, then $\sqrt{a}$ is

(a) real (b) integer (c) irrational (d) rational

(vi) If a +b = a, what is value of b?

(a) 1 (b) –1 (c) a (d) 0

(vii) If a.b = 1, what is value of b?

(a) 1 (b) $\frac{1}{b}$ (c) $\frac{1}{a}$ (d) –1

(viii) According to reflexive property : $y^2 - 17 = ?$

(a) y^2+17 (b) $y - 17$ (c) $y^2 - 17$ (d) $-17 - y^2$

(ix) If a.b = a, what is value of b?

(a) $\frac{1}{a}$ (b) 1 (c) a (d) -1

(x) If a.b = 1, what is b called?

(a) multiplicative inverse of a (b) additive identity

(c) multiplicative identity (d) self-multiplicative inverse

(xi) Commutative property does not hold with respect to:

(a) addition (b) multiplication

(c) subtraction (d) both (a) and (b)

2. Represent each number on the number line.

(i) $-5\frac{1}{5}$ (ii) $\frac{17}{3}$ (iii) $-2 < x < 4$ (iv) $x \geq 6$

3. Write each exponential expression as an equivalent radical expression and simplify if possible.

(i) $(-2)^{\frac{4}{5}}$ (ii) $(-27)^{\frac{1}{3}}$ (iii) $(\sqrt{16})^4$

(iv) $(\sqrt[3]{-8})^9$ (v) $(x^{-2})^3.(x^0)^5$

4. Use the properties of exponents to simplify each of the following.

(i) $\frac{(-2)^3.(-2)^{-4}.(-2)}{(-2)^{-3}}$ (ii) $\frac{2^{\frac{1}{2}}.2^{\frac{3}{4}}}{2^{\frac{1}{2}}} \times \frac{3.3^{\frac{3}{2}}}{3^{-\frac{1}{2}}}$

5. Determine whether each statement is true or false. If false, give an example of a number that shows the statement is ture.

a. Every rational number is an integer.

b. Every real number is an irrational.

c. Every irrational number is a real number

d. Every integer is a rational number.

e. Every real number is either rational number or an irrational number.

UNIT
02

LOGARITHMS

In this unit the students will be able to:

- Express a number in standard form of scientific notation and vice versa.
- Define logarithm of a number to the base *a*.
- Define a common logarithm, characteristic and mantissa of log of a number.
- Use tables to find the log of a number.
- Give concept of antilog and use tables to find the antilog of a number.
- Differentiate between common and natural logarithm.
- Prove the four basic laws of logarithm.
- Apply laws of logarithm to convert lengthy processes of multiplication, division and exponentiation into easier processes of addition and subtraction etc.

October 8, 2005 is an unforgettable day in the history of Pakistan, when the earth started shaking violently and in few minutes the worst disaster had ruined many of the towns and villages from the face of the earth. This earthquake measured 7.6 on Richter scale but what is a Richter scale? Answer to this question will be explained in this unit.

INTRODUCTION

Exponents provide an efficient way of writing very large as well as very small numbers. For example: approximate mass of Uranus is 87 trillion trillion kg i.e. 87 followed by 24 zeros or 87, 000000000000, 000000000000.

This style of expressing a number is called *standard form* which is not useful for such a large number, since some error may occur while writing or telling it. There is another method of writing such numbers, to make them handy.

This method involves integral exponents of 10. In this method the mass of Uranus is $8.7\times10{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$ kg = 8.7×10^{25} kg. This method is called Scientific notation.

Historya Mystery

Al-Khawarizmi did pioneering work on logarithms and the word **logarithm** is also derived from his name.

2.1 Scientific Notation

A number 'c' is in scientific notation if it is written as $c = d\times10^n$, where $1\le d<10$ and $n\in Z$.

For example: 5.3×10^7, 7.412×10^{-2}, 1.592×10^0.

How to Write in Scientific Notation

- Place the decimal point after first left hand nonzero digit, the resulting number is d. (Position after first left hand nonzero digit is called **reference position**.)
- Count the number of digits moved by the decimal point. This is absolute value of n.
- If decimal point is moved to left, value of n is positive.
- If decimal point is moved to right, the value of n is negative.

e.g. $0.05\,432 = 05.432\times10^{-2}$ or 5.432×10^{-2}

and $5\,43.2 = 5.432\times10^2$

Key Fact

Decimal point is at the right of last digit in an integer.

Example 1: Convert the following into scientific notation:

(a) One light year: 5880,000,000,000 miles

$5\,880000000000.0 = 5.88\times10^{12}$

The number is greater than 10 so exponent must be positive.

(b) Mass of the smallest insect = 0.00000492 g

$0.000004\,92 = 4.92\times10^{-6}$ g

The number is smaller than 1 so exponent must be negative.

Standard Notation: The number already written in scientific notation, can be converted to standard notation by the multiplication of its two factors.

Example 2: Convert the followings into standard notation.

(a) Density of hydrogen = 8.99×10^{-5} g/cm^3

$8.99 \times 10^{-5} = 0\ 00008\,.99 \times 10^{-5} = 0.0000899$

> Exponent is negative so the number is smaller than 1.

> Exponent is positive so the number will be greater than 10.

(b) Number of air sacs in lungs = 3.5×10^8

$3.5 \times 10^8 = 3\,.50000000\,0 \times 10^8 = 350000000.0$ or 350000000

Example 3: The closest star to the Earth (other than Sun) is Alpha Centauri, 4.35 light years from Earth. How many kilometers from Earth is Alpha Centauri?

If one light year = 9460920 million km. Write the answer in scientific notation.

Solution: One light year = 9460920×10^6 km

Distance between Earth and Alpha Centauri

$= 4.35$ light years $= 4.35 \times 9460920 \times 10^6$ km

$= 41155002 \times 10^6 = 4.1155002 \times 10^6 \times 10^7$

$= 4.1155002 \times 10^{13}$ km

> **Calculator Site**
>
> Most of the calculators have a key **E** or **EXP**, For entering a number in scientific notation.

Example 4: The speed of light is approximately 3×10^5 km/s and distance between earth and sun is approximately 1.5×10^8 km. If the sun is suddenly to burn out, how long would it take for earthlings to know about it? Write the answer in standard notation.

Solution: Formula for finding time, if the speed and distance are given, is

Time = Distance/Speed

Here, speed of light = 3×10^5 km/s and distance between Earth and Sun $=1.5\times10^8$km.

$$\text{Time} = \frac{1.5\times10^8\,km}{3\times10^5\,km/s} = 0.5 \times 10^{8-5}\text{ sec} = 0.5 \times {}^{3}10\text{ sec} = 500\text{ sec or } 8\text{ min } 20\text{ sec}$$

EXERCISE 2.1

1. Write the following in scientific notation.
   (i) 0.00053407 (ii) 53400000 (iii) 0.000000000012 (iv) 2.5326
2. Write the following in standard notation.
   (i) 9.067×10^{-5} (ii) 5.64×10^0 (iii) 6.53×10^{-6} (iv) 3.1415×10^9
3. Simplify the following by converting into the form indicated.
   (i) $563.71 \times 10^{-3} \times 2.54 \times 10^4$ ⟶ scientific notation
   (ii) $\dfrac{0.023\times10^5}{10^{-3}}$ ⟶ standard notation
   (iii) $\dfrac{2.549\times5067\times10^{-3}}{10^3}$ ⟶ scientific notation
   (iv) 0.0009988×10^{10} ⟶ standard notation

4. If it takes 5 seconds to recite 'Kalma Pak' once, how many hours will it take to recite 'Kalma Pak' one million times? Convert hours into days and write the answer in standard form. Round off the answer, discarding the decimal part.
5. Distance between Earth and Sun is 9.3225600×10^{7} miles. If speed of light is approximately $1.86,000\times10^{5}$ miles per second, how long does it take for light to reach the Earth. Convert the answer in minutes writing in standard form.

2.2 Logarithms

2.2.1 Why We Use Logarithms

Population of the world is growing and the radioactive wastes are decaying continuously. The mathematical tool used to predict the future and explore the past of such rates of growth and decay over time, is an exponential relation.
i.e. an equation of the form $x = b^{y}$ where b, x and y are real numbers, $b > 0$, $x > 0$ and $b \neq 1$. This relation is widely used by archaeologists, scientists and business people. The inverse relation of this exponential relation is called logarithmic relation.

Key Fact

In late 1500s, John Napier extended the work of Al–Khawarizmi and started developing log tables.

Definition of Logarithm

If $b^{y} = x$ where $x, y, b \in R$; $b > 0, x > 0$ and $b \neq 1$, then y is the **logarithm** of x with base b, written as $y = \log_b x \Leftrightarrow b^{y} = x$.

While evaluating logarithms, remember that a logarithm is an exponent, e.g. if $\log_9 81 = 2$, then 2 is the logarithm of 81 with base 9, since 9 raised to power 2 gives 81.

Example 5: Convert the following exponential equations to logarithmic equations and the logarithmic equations to exponential equations.

(a) $2^{7} = 128$, here base = 2, exponent = 7 and $x = 128$

exponent

x

$$2^{7} = 128 \Leftrightarrow \log_2 128 = 7$$

Base

Enlighten Yourself

Exponential equations are used by
a. Archaeologists, for finding the age of very old bones, fossils etc.
b. Scientists for finding the life time of radioactive elements etc.

(b) $$7^{-3} = \frac{1}{343} \Leftrightarrow \log_7 \frac{1}{343} = -3$$

(c) $\sqrt[3]{125} = 5$ or $(125)^{\frac{1}{3}} = 5$

$$125^{\frac{1}{3}} = 5 \Leftrightarrow \log_{125} 5 = \frac{1}{3}$$

(d) $\log_5 625 = 4 \quad \Leftrightarrow \quad 5^4 = 625$

(e) $\log_2 \dfrac{1}{64} = -6 \quad \Leftrightarrow \quad 2^{-6} = \dfrac{1}{64}$

(f) $\log_{81} \dfrac{1}{3} = -\dfrac{1}{4} \quad \Leftrightarrow \quad (81)^{-\frac{1}{4}} = \dfrac{1}{3}$

Check Point

$\text{Log}_t 5^0 = ?$

$\text{Log}_3 (\log_2 2) = ?$

Key Fact

- $\log_b x$ is defined only for positive x.
- $\log_b 1 = 0 \quad \because b^0 = 1$
- $\log_b b = 1 \quad \because b^1 = b$
- $\log_a a^x = x \quad \because a^x = a^x$
- $\log_b x_1 = \log_b x_2 \quad \Rightarrow x_1 = x_2$

Example 6: Check whether these logs are defined or not?

(a) $\log_1 2 = y \quad \Rightarrow \quad 1^y = 2$

None of the exponents of 1 can give answer 2, so $\log_1 2$ is undefined.

(b) $\log_5 (-1) = y \quad \Rightarrow \quad 5^y = -1$

None of the exponents of 5 can give answer '–1', so $\log_5 (-1)$ is undefined i.e. log of negative number is not defined.

(c) $\log_2 0 = y \quad \Rightarrow \quad 2^y = 0$

None of the exponent of 2, can give answer 0, so log of 0 is not defined.

(d) Is $\log_2 (4 - 2x) = y$ true or not for $x = 0, 1, 2$.

$\log_2 (4 - 2x) = y \quad \Rightarrow \quad 2^y = 4 - 2x$

If $x = 0$, then $2^y = 4$ is true for $y = 2$.

If $x = 1$, then $2^y = 2$ is true for $y = 1$.

If $x = 2$, then $2^y = 0$ is not true for any value of y.

Example 7: Find the value of unknowns by converting logarithmic form to exponential form.

(a) $\log_2 x = 4 \quad \Rightarrow \quad 2^4 = x \quad \Rightarrow \quad x = 2 \times 2 \times 2 \times 2 = 16$

(b) $\log_{64} x = -\dfrac{4}{3} \Rightarrow (64)^{-\frac{4}{3}} = x \quad \Rightarrow \quad x = (4^3)^{-\frac{4}{3}} = 4^{3 \times -\frac{4}{3}} = \dfrac{1}{4^4} = \dfrac{1}{256}$

(c) $\log_b \dfrac{1}{128} = -7 \quad \Rightarrow \quad b^{-7} = \dfrac{1}{128} = \dfrac{1}{2^7}$ or $b^{-7} = 2^{-7}$

As exponents are same, so bases must be same i.e $b = 2$

(d) $\log_{27} 3 = y \quad \Rightarrow \quad 27^y = 3$ or $(3^3)^y = 3 \quad \Rightarrow \quad 3^{3y} = 3^1$

As bases are same so exponents must be same. i.e. $3y = 1$ or $y = \dfrac{1}{3}$

Check Point

Decide which log is defined:

Log 1	Log 0
Log–2	$\text{Log}_{-10}2$

Example 8: Find y if $\log_b (y^3 + 1) = \log_b 28$

Solution: $\log_b (y^3 + 1) = \log_b 28$

$\Rightarrow y^3 + 1 = 28$ or $y^3 = 28 - 1 = 27$

$y = \sqrt[3]{27} = 3$

EXERCISE 2.2

1. Check whether $\log_x(7-x)$ is defined for
   (i) $x=0$ **(ii)** $x=1$ **(iii)** $x=6$ **(iv)** $x\geq 7$
2. Convert the form of following equations i.e. from exponential form to logarithmic form and vice versa
   (i) $\log_6 216=3$ **(ii)** $7^4=2401$ **(iii)** $\log_5 x=5$ **(iv)** $b^{-\frac{3}{4}}=\frac{1}{27}$
   (v) $125^{\frac{x}{3}}=25$ **(vi)** $\log_{10}10^{12}=y$ **(vii)** $(256)^{\frac{x}{4}}=\frac{1}{64}$
   (viii) $\log_3(x^3+1)=2$ **(ix)** $\log_5(2x-3)=1$ **(x)** $2x+1=2^3$
3. Find the value of x in the following questions.
   (i) $\log_x 3=1$ **(ii)** $\log_{x+1}9=2$ **(iii)** $\log_3 81=x$
   (iv) $\log_2 64=x+1$ **(v)** $\log_2 x=4$ **(vi)** $\log_2(x^2-1)=3$
4. Find the unknowns appeared in the question 2.

2.2.2 Common Logarithm

There are two most commonly used bases for logarithm i.e.'10' and 'e ≈ 2.71828'(an irrational number). Base 10 was used by Henry Briggs.

*If $10^y=x$, for $x>0$, then y is called **common log** of x i.e. $10^y=x \Leftrightarrow \log_{10}x=y$*

These logarithms are also called Briggs's logarithms, denoted by $\log_{10}x$ or simply $\log x$. If none of the base is mentioned with log then it is obviously a common logarithm e.g. $\log_{10}36$ can be simply written as log 36.

Logarithm of a number = Characteristic + Mantissa

2.2.3 Characteristic

*Integral part of the logarithm is called **characteristic.***

Characteristic is an integer. It is infact the integral power of 10, when the number is written in scientific notation. e.g characteristic of $\log 3.3\times 10^2$ is '2' and in $\log 5.632\times 10^{-4}$, characteristic is negative 4. This negative characteristic is usually written as $\bar{4}$.

Characteristic of the log of some number can also be found using **reference position**. In 0.00532, the reference position is between 5 and 3. By counting the number of digits between the decimal point and the reference position we get the numerical value of the characteristic however, the sign is taken negative if the reference position is on right side of the decimal point and it is taken positive otherwise.

Memory Plus

If $\log x_1=\log x_2 \Rightarrow x_1=x_2$
- $\log 10^0=0$
  $\log 1=0$
- $\log 10^1=1$
- $\log 10^2=2$
- $\log 10^3=3$
- $\log 10^{12}=12$

So log of an integral power of 10 is that integral whole number,
- lso If $1<x<10$, then $0<\log x<1$.

2.2.4 Mantissa

Decimal part or the fractional part of a logarithm is called mantissa.

Mantissa is always a nonnegative number less than 1, i.e. it can be either zero or positive but never negative. Mantissa is found from the log table.

A small part of log table:

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 36
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 13	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	19 17	22 20	25 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 6	9 8	11 11	14 14	16 17	20 19	23 22	26 24

Example 9: Find **(a)** log 156.3 **(b)** log 0.0123

Solution: (a) $\log 156.3 = \log 1.563 \times 10^2$ — Convert the number into scientific notation

characteristic = 2

Explanation: Llook at the log table in the extreme left column for the number 15. The next digit in 156.3 is 6 From the top row, look at the digit 6. Move vertically downward from 6 and horizontally rightwards from 15. The number present at the intersection of row of 15 and the column of 6 is 1931. Go ahead horizontally and see the number present at the intersection of row of 15 and column of 3 (in the difference tables) i.e. 8. Add 1931 and 8 to get 1939. Since mantissa is less than 1, so mark the decimal point before first digit so mantissa is '.1939'.

mantissa = .1939 or 0.1939

log 156.3 = characteristics + mantissa = 2 + 0.1939 = 2.1939

(b) $\log 0.0123 = \log 1.230 \times 10^{-2}$

characteristic = –2 or $\bar{2}$

For mantissa, use the log table to see the number present at the intersection of row of 12 and the column of 3 i.e. 0899, as there is no difference table for '0' so mark the decimal point before the first digit i.e. mantissa is .0899.

mantissa = .0899

log 0.0123 = $\bar{2}$ + .0899 = $\bar{2}$.0899 (never write '–2.0899')

Example 10: Find log 1009

Solution:

$\log 1009 = \log 1.009 \times 10^3$

characteristic = 3

mantissa = .0038 ($\neq$ .38)

$\therefore$ log 1009 = 3.0038

10| 0|9

.0000
+.0038
.0038

Memory Plus

Number of digits in a whole number = characteristics +1

i.e If characteristic of log of some whole number is 3, then the number of digits in that number will be

3 +1=4

see Example 10 for confirmation.

EXERCISE 2.3

Find the logarithms of following numbers if possible.

1. 5313	2. 4580	3. 9.613	4. 110.9
5. 52.39	6. 0.01207	7. 0.0093	8. 1 Trillion
9. 0 .00004	10. 4	11. 4000	12. 54
13. 1.009	14. 0.1009	15. – 4	

2.2.5 Antilogarithm

The inverse operation of taking log is called antilog. An antilog is used to cancel the effect of log.

If $\log x = y$ *then x is called the* ***antilog*** *of y* or $x = \text{antilog } y$.

Antilogarithm tables are used for finding it. Before finding antilog, recall that log x consists of two parts, characteristic and mantissa. For finding antilog, mantissa is used for looking in the antilog table. Characteristic is not used in table, however characteristic is used for locating the decimal point in the number obtained from antilog table.

A small part of Antilog of Table:

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Example 11(a): Find antilog 2.4541

Solution: 2.4541 is obviously log of some number so,

characteristic = 2 (integral part) , mantissa = .4541 (fractional part)

Explanation: Look at the antilog table for .45 in extreme left column and 4 (the digit next to .45) in the top row. The number present at the intersection of row of .45 and column of 4 is 2844. Now go ahead horizontally, the number present at the intersection of row of .45 and difference column of 1 is 1. Add 2844 and 1 to get 2845 . Antilog table is no more needed, however the antilog of 3.4541 is not yet completely found. Locate the reference position in the number obtained from antilog table i.e. 2 ∧ 845, now characteristic will locate the decimal point . As characteristic is +2, so mark the decimal point moving two digits rightwards from reference position i.e. 2 ∧ 84 . 5

$\therefore$ antilog 2.4541 = 284.5

(b) If $\log x = \bar{2}.0000$, then find x.

Solution: $\log x = \bar{2}.0000$

$\text{antilog}(\log x) = \text{antilog}(\bar{2}.0000)$ (taking antilog on both sides.)

$x = \text{antilog}(\bar{2}.0000)$

here, characteristic = $\bar{2}$ and mantissa = .0000 (see log table at the back of this book.)

$\therefore\ x = \text{antilog}\,(\bar{2}.0000)$

$= .01000$

Example 12: Find antilog of $\bar{2}.4900$

Solution: antilog $\bar{2}.4900$

characteristic = $\bar{2}$ mantissa = .4900

antilog $(\bar{2}.49)$ = $.03 \wedge 090 = 0.03090$

Rough work for antilog

. 49 ↓0 ↓0

3090 →

EXERCISE 2.4

Find the antilog of following numbers.

1.	2.4324	2.	1.5890	3.	0.2425	4.	3.5636
5.	4	6.	0.0038	7.	$\bar{1}.2429$	8.	$\bar{2}.9281$
9.	$\bar{3}.5219$	10.	0.0000	11.	–3	12.	5.9990
13.	2.4900	14.	0.49000	15.	2.34		

2.3 Common and Natural Logarithm

John Napier started developing log tables with base e, so the logarithms with base e are called natural logarithms or Naperian logarithms, represented by '$\ln x$'.

2.3.1

*If $e^y = x$, for positive values of x then y is called **natural log** of x i.e. $e^y = x \Leftrightarrow y = \ln x$.*

Napier spent last 20 years of his life working with log tables of base e, which he never finished and died. Henry Briggs, then completed these tables. The difference between common and natural logarithms is depicted below.

Log Type	Representation	Base	Nature of base	Properties
Common (Briggs)	$\log x$	10	Rational	$\log 1 = 0$ $\log 10 = 1$ $\log 10^x = x$
Natural (Naperian)	$\ln x$	$e \approx 2.71828$	Irrational	$\ln 1 = 0$ $\ln e = 1$ $\ln e^x = x$

2.4 Laws of Logarithms

Laws of logarithms are closely related to the laws of exponents, since logarithms in nature, are exponents. The laws of exponents are given in unit 1. In this section the laws of exponents are used to develop some laws of logarithms, for solving complicated equations involving exponents and logarithms. Also the complicated questions of multiplication, division and root extraction are converted to handy sums of addition and subtraction.

2.4.1 Product Law of Logarithms

For any real numbers m, n and b where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$

Proof: Let $\log_b m = x$ .... (i) and $\log_b n = y$ ...... (ii)

Their respective exponential equations are

$m = b^x$ ...... (iii) and $n = b^y$ ...... (iv)

Now product of equations (iii) and (iv) is

$m \times n = b^x . b^y$

$mn = b^{x+y}$ ← using product rule of exponents

$\Rightarrow \log_b mn = x + y$ ←logarithmic form of above equation

or $\log_b mn = \log_b m + \log_b n$ ← substituting the values of x and y

The Product law of logarithms states that:

Logarithm of a product of two (or more) numbers is equal to the sum of their logarithms, provided that all logarithms are defined.

Key Fact

$\log_{10}(3 \times 2) = \log_{10}3 + \log_{10}2$

Example 13 (a): Use the product rule to expand $\log_e(x^2 y^3 z)$.

Solution $\log_e(x^2 y^3 z) = \log_e x^2 + \log_e y^3 + \log_e z$

(b) Use product rule to combine $\ln a + \ln\sqrt{b} + \ln c^3$ in a single logarithmic term.

Solution $\ln a + \ln\sqrt{b} + \ln c^3 = \ln(a\sqrt{b}\,c^3)$

2.4.2 The Quotient Law of Logarithm

For any positive numbers m, n and b, where $b \neq 1$.

$$\log_b \frac{m}{n} = \log_b m - \log_b n$$

Proof: Let $\log_b m = x$ ...... (i) and $\log_b n = y$...... (ii)

Respective exponential forms of above equations are

$m = b^x$ ...... (iii) and $n = b^y$ ...... (iv)

Dividing (iii) by (iv) i.e. $\frac{m}{n} = \frac{b^x}{b^y}$

$\frac{m}{n} = b^{x-y}$ ←quotient rule of exponents with same base

$\Rightarrow \log_b \frac{m}{n} = x - y$ ←conversion to logarithmic form

$\log_b \frac{m}{n} = \log_b m - \log_b n$ ←Substituting the value of x and y

The Quotient law of logarithms states that:

Logarithms of quotient of two numbers is equal to the difference of their logarithms provided that all the logarithms are defined.

Example 14 (a): Use quotient law of logarithms to expand $\log \dfrac{13}{7b}$.

(b) Use quotient law of logarithms to combine log 39 – log t – log b into single logarithmic term.

Solution (a): $\log \dfrac{13}{7b} = \log 13 - (\log 7 + \log b) = \log 13 - \log 7 - \log b$

Solution (b): $\log 39 - \log t - \log b = \log 39 - (\log t + \log b) = \log 39 - \log tb = \log \dfrac{39}{tb}$

Key Fact

$\log(50 - 5) \neq \log 50 - \log 5$

2.4.3 The Power Law of Logarithm

For any real number m, n and b, where $m > 0$, $b > 0$, $b \neq 1$, $\log_b m^n = n \log_b m$

Proof: Let $\log_b m = x$

Converting into exponential equation i.e $m = b^x$

$m^n = (b^x)^n$ ← taking n^{th} power on both sides.

or $m^n = b^{nx}$ ← using power of power rule .

$\Rightarrow \log_b m^n = nx$ ← converting into logarithmic form .

$\log_b m \ = n \log_b m$

← substituting the value of x.

The Power Law of logarithms states that:

Logarithm of power of a number is equal to the power times the logarithm of the number, provided that all logarithms are defined.

Example 15(a): Use power law of logarithms to expand $\log_2 100^{-3}$.

Solution: $\log_2 100^{-3} = -3 \log_2 100$

(b) Use power law of logarithms to combine $-\dfrac{4}{3}\log_{\sqrt{3}} 7$

Solution: $-\dfrac{4}{3}\log_{\sqrt{3}} 7 = \log_{\sqrt{3}} 7^{-\frac{4}{3}}$

2.4.4 Change of Base Law of Logarithms

Although only common logarithms and natural logarithm are programmed into a calculator still the logarithms for other positive real bases can be found by changing that base into some frequently used base that is 'e' or '10'. The law which enables us to change the base is called change of base law which is given below.

If a, b and m are positive real numbers and, $a \neq 1$, $b \neq 1$, then

$\log_a m = \log_b m \cdot \log_a b$

Proof: Let $\log_b m = x$

$\Rightarrow \quad m = b^x$ ←converting above equation into exponential equation.

$\log_a m = \log_a b^x$ ←taking log on both sides with base 'a'.

$\log_a m = x.\log_a b$ ←applying power law of logarithm.

$\log_a m = \log_b m.\log_a b$ or $\log_b m = \dfrac{\log_a m}{\log_a b}$

Slide Rule

based upon laws of logarithms were used for complicated calculations, before the invention of calculators.

Example 16: Convert the base of $\log_b 536$ into 10.

Solution: $\log_b 536 = \dfrac{\log_{10} 536}{\log_{10} b}$ or $\dfrac{\log 536}{\log b}$

Key Fact

The change of base law and the quotient law of logarithms are often confused. Remember the difference between $\log_b\left(\dfrac{m}{n}\right)$ and $\dfrac{\log_a m}{\log_a n}$. These two expressions look alike, however they are totally different.

Example 17(a): Use laws of logarithms to expand $\log \dfrac{5p^2q^{\frac{1}{2}}}{4\sqrt{s}t^3}$.

Solution: $\log \dfrac{5p^2q^{\frac{1}{2}}}{4\sqrt{s}t^3} = \log(5p^2q^{\frac{1}{2}}) - \log(4\sqrt{s}\ t^3)$ ←using quotient law of log

$= \log 5 + \log p^2 + \log q^{\frac{1}{2}} - [\log 4 + \log\sqrt{s} + \log t^3]$ ←using product law of log

$= \log 5 + 2\log p + \dfrac{1}{2}\log q - \log 4 - \dfrac{1}{2}\log s - 3\log t$ ←using power law of log

(b) Use laws of logarithms to evaluate

(i) $\log_2 5\sqrt{3}$ **(ii)** $\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

Solution: **(i)** $\log_2 5\sqrt{3} = \dfrac{\log_{10} 5\sqrt{3}}{\log_{10} 2}$ ← change of base law

$= \dfrac{\left[\log 5 + \log 3^{\frac{1}{2}}\right]}{\log 2} = (\log 5 + \dfrac{1}{2}\log 3) \div \log 2$

$= [0.6990 + \dfrac{1}{2}(0.4771)] \div (0.3010)$

$= \dfrac{0.9376}{3.010} = 3.1148$

(ii) Evaluate $\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

$\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

$$=\log_2\left(\log_2(2^3)^2-\log_{\sqrt{3}}\left(\left(\sqrt{3}\right)^2\right)^3+\log_{\sqrt{10}}\left(\sqrt{10}\right)^2\right)$$

$$=\log_2\left(\log_2 2^6-\log_{\sqrt{3}}\left(\sqrt{3}\right)^6+\log_{\sqrt{10}}\left(\sqrt{10}\right)^2\right)$$

$$=\log_2\left(6\log_2 2-6\log_{\sqrt{3}}\sqrt{3}+2\log_{\sqrt{10}}\sqrt{10}\right) \qquad (\because \log_b b^x = x)$$

$$=\log_2(6-6+2)$$

$$=\log_2 2 = 1 \qquad (\because \log_b b = 1)$$

c. If $\log_b 2 = 0.3010$, $\log_b 3 = 0.4771$ and $\log_b 5 = 0.6990$, then evaluate $\log_b 0.0036$, applying laws of logarithms.

Math Play Ground

Maths play ground
1. Take students to the play ground.
2. Give each student a strip of paper with a simple logarithms sum written on it.
3. Spread answers of all questions in the play ground and ask students to find their respective answers.
4. Specimen questions may include: (i) log10 = ? (ii) lne = ? etc.

Solution: $\log_b 0.0036 = \log_b\left(\frac{36}{10000}\right)$

$$=\log_b\left(\frac{2^2\times3^2}{2^4\times5^4}\right)$$

$$=\log_b\left(\frac{3^2}{2^2\times5^4}\right)$$

$$=2\log_b 3-2\log_b 2-4\log_b 5$$

$$=2(0.4771)-2(0.3010)-4(0.6990)$$

$$=-2.4438$$

(which is negative number having both characteristic and mantissa as negative.)
Since mantissa can never be negative, to make the mantissa positive check the next positive integer to the magnitude of the answer. The next integer is '+3' as shown on the number line

Now 'add 3 to' and 'subtract 3 from' the answer
i.e. $\log_b 0.0036 = -2.4438 + 3 - 3 = 0.5562 - 3$.
The positive term is mantissa and negative term is characteristic.
So, $\log_b 0.0036 = \bar{3}.5562$

Important results deduced from Laws of Logarithms

(i) $\log_b a \times \log_a b = 1$	(ii) $\log_c a = \frac{1}{\log_a c}$
(iii) $\log_b a \,.\, \log_c b = \log_c a$	(iv) $\log_s r^t \times \log_t s^r \times \log_r t^s = rst$
(v) $\log_x z \times \log_y x \times \frac{1}{\log_y z} = 1$	(vi) $\log_b a \,.\, \log_c b \times \log_a c = 1$

EXERCISE 2.5

1. Use laws of logarithms to expand the followings.

(i) $\log 9t$ (ii) $\log \frac{59}{s}$ (iii) $\log \frac{(5pq^2)}{(xy^3)}$ (iv) $\log \sqrt{\frac{53.3}{46.4}}$

(v) $\log \left(\frac{5^2 t^5 a^{\frac{1}{3}}}{\sqrt[3]{4.4tb^3}}\right)$ (vi) $\log \sqrt[5]{\frac{7^2 t^3 p}{d^6 b^2}}$ (vii) $\log \left(\frac{\sqrt[3]{5.512}\, pm^{\frac{1}{2}}}{\sqrt[4]{5.91}\, a^2 b}\right)^t$

2. Use laws of logarithms to combine the followings into single logarithmic terms.

(i) $3\log x - 5\log y$

(ii) $\frac{1}{2}\log t + \frac{1}{3}\log r - \frac{1}{5}\log s$

(iii) $\frac{1}{7}[\log 57.7 - 3\log 9.24 + 4\log 36.6 - 2\log 23.3]$

(iv) $5\log 6 - 7\log 9.42 + \frac{1}{3}\log t - \frac{1}{2}\log 32.2 + \frac{2}{3}\log a$

(v) $\frac{5}{4}\log 37.74 - \frac{1}{4}\log 53.71 + \frac{1}{4}\log 28.83$

3. Use laws of logarithms to evaluate the followings.

(i) $\log_2 15$ (ii) $\log_9 \sqrt[3]{9}$ (iii) $\log_3 65$ (iv) $\log_{\sqrt{3}} 72.34$ (v) $\log_{\sqrt{7}} 343$

4. If $\log_b 2 = 0.3010$, $\log_b 3 = 0.4771$, $\log_b 5 = 0.6990$ then evaluate the followings with laws of logarithms.

(i) $\log_b \frac{6}{5}$ (ii) $\log_b \frac{100}{9}$ (iii) $\log_b \frac{\sqrt[3]{450}}{\sqrt{27}}$ (iv) $\log_b 0.024$ (v) $\log_b \sqrt[7]{5\frac{2}{5}}$

2.5 Applications of Logarithms

Common logarithms appear in many scientific formulae.

Richter Scale

Opening of this unit depicts the 2005 earthquake in Pakistan, when thousands of people lost their lives. The Richter Scale used for measuring the magnitude (M) of earthquake is a logarithmic scale. If I is intensity of its shock waves and I_0 is a constant then

$$M = \log\left(\frac{I}{I_0}\right)$$

Example 18: An earthquake that occurred in Pakistan in 2005, measured 7.6 on the Richter scale. In 1978, an earthquake in China measured 8.2 on the Richter scale. How many times, the China's earthquake was stronger than the Pakistan's earthquake.

Solution: Let I_1 be the intensity of Pakistan's earthquake and I_2 be the intensity of China's earthquake.

As, $M = \log\left(\frac{I}{I_0}\right)$

Then, $7.6 = \log\left(\frac{I_1}{I_0}\right) = \log I_1 - \log I_0$ .... (i)

$8.2 = \log\left(\frac{I_2}{I_0}\right) = \log I_2 - \log I_0$ .... (ii)

Project

Search more applications of logarithms in Biology, Chemistry and Physics.

Subtracting (i) from (ii)

$\log I_2 - \log I_1 = 8.2 - 7.6$

$\Rightarrow \quad \log\left(\frac{I_2}{I_1}\right) = 0.6$

$\text{antilog}\log\left(\frac{I_2}{I_1}\right) = \text{antilog } 0.6$ (taking antilog on both sides)

$\frac{I_2}{I_1} = 3.98 \approx 4$

$\therefore$ China's earthquake is nearly 4 times stronger than Pakistan's earthquake.

Example 19: Use laws of logarithms to evaluate.

(a) $\frac{\sqrt[3]{8.59}\times(55.6)^2}{2.51\times\sqrt{2.12}}$ **(b)** $\sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$

Solution: **(a)** let $x = \frac{\sqrt[3]{8.59}\times(55.6)^2}{2.51\times\sqrt{2.12}}$

Then, by taking common log on both sides.

$$\log x = \log\frac{\sqrt[3]{8.59}\times(55.6)^2}{2.51\times\sqrt{2.12}}$$

$$= \log(8.59)^{\frac{1}{3}} + \log(55.6)^2 - \log(2.51) - \log(2.12)^{\frac{1}{2}}$$

$$= \frac{1}{3}\log 8.59 + 2\log 55.6 - \log 2.51 - \frac{1}{2}\log(2.12)$$

$$\approx \frac{1}{3}(0.9340) + 2(1.7451) - (0.3997) - \frac{1}{2}(0.3263)$$

$$\approx 0.3113 + 3.4902 - 0.3997 - 0.1632$$

$\log x \approx 3.2386$

$\text{antilog}\log x \approx \text{antilog } 3.2386$ ← Taking antilog on both sides

$x \approx 1732.21$

$\therefore \frac{\sqrt[3]{8.59}\times(55.6)^2}{2.51\times\sqrt{2.12}} \approx 1732.21$

(b) let $x = \sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$

$\log x = \log \sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$ ← Taking common log on both sides

$\log x = \log \left(\frac{16}{3}\right)^{\frac{1}{3}} - \log \left(\frac{22}{7}\right)^{\frac{1}{2}} = \frac{1}{3}\log\left(\frac{16}{3}\right) - \frac{1}{2}\log\left(\frac{22}{7}\right)$

$= \frac{1}{3}(\log 16 - \log 3) - \frac{1}{2}(\log 22 - \log 7)$

$= \frac{1}{3}\log 16 - \frac{1}{3}\log 3 - \frac{1}{2}\log 22 + \frac{1}{2}\log 7$

$\approx \frac{1}{3}(1.2041) - \frac{1}{3}(0.4771) - \frac{1}{2}(1.3424) + \frac{1}{2}(0.8451)$

$\log x \approx 0.4014 - 0.1590 - 0.6712 + 0.4226$

$\log x \approx -0.0062$

$\log x \approx (-0.0062 + 1) - 1$ ← making mantissa positive

$\approx 0.9938 - 1$

$\log x \approx \bar{1}.9938$

Taking antilog on both sides.

antilog $\log x \approx$ antilog $\bar{1}.9938$

$x \approx 0.9858$

Example 20: Find the number of digits in 5^{50}.

Solution:

By finding the log of the given whole number, we can find the relevant characteristic. But the number of digits in a whole number is always one more than the characteristic of the log of that number.

Now $\log 5^{50} = 50 \log 5$

$= 50 \times (.6990) = 34.95$

Characteristic = 34

Number of digits in 5^{50} = characteristic $+ 1 = 34 + 1 = 35$

EXERCISE 2.6

1. Find the number of digits in

   (i) 3^{30} (ii) 100^{100} (iii) 2^{10} (iv) 5^{37} (v) 529^{30} (vi) 23^{15}

2. Evaluate applying laws of logarithms.

   (i) 23.57×5.967 (ii) $\dfrac{65.89}{7.392}$ (iii) $\dfrac{47.27 \times 5.321}{9.712 \times 4.171}$ (iv) $\dfrac{\sqrt[3]{27.98}}{\sqrt[2]{28.73}}$

   (v) $\dfrac{\sqrt[7]{129.4}}{\sqrt[3]{27.37}}$ (vi) $\dfrac{\sqrt{39.24} \times \sqrt[3]{1.931}}{\sqrt[4]{64.4} \times \sqrt{23.91}}$ (vii) $\dfrac{\sqrt{16\frac{3}{4}}}{\sqrt[3]{53}}$

   (viii) $\dfrac{(27.98)^2}{(28.73)^3}$

3. The Kansu, China earthquake of 1920 was measured about 8.5 on Richter Scale and the Tokyo, Japan earthquake of 1923 was measured 7.8 on that scale how many times stronger was the 1920 earthquake than 1923 earthquake?

KEY POINTS

- A number written as $d \times 10^n$,(where $1 \leq d < 10$, $n \in Z$) is said to be in Scientific notation.
- Reference position is the place after first left hand non–zero digit.
- For x, y, $b \in R$; $b > 0$, $x > 0$ and $b \neq 1$, y is called logarithm of x with base b written as

$$y = \log_b x$$

- Logarithm of a number consists of two parts,
  - Characteristic: The integral part of logarithm.
  - Mantissa: The fractional part of logarithm, which is never negative.
- If $\log x = y$ then x is called the antilog of y.
- $y = \log_{10} x$, is called common logarithm and $y = \log_e x$ is called natural logarithm.
- $\log_b mn = \log_b m + \log_b n$
- $\log_b \dfrac{m}{n} = \log_b m - \log_b n$
- $\log_b m^n = n \log_b m$
- $\log_n m = \dfrac{\log_b m}{\log_b n}$

MISCELLANEOUS EXERCISE 2

1. Encircle the correct option in the following.

(i) If $a = b \times 10^n$ is written in scientific notation then ,

(a) $0 \le b \le 10$ **(b)** $0 \le b < 10$ **(c)** $1 \le b \le 10$ **(d)** $1 \le b < 10$

(ii) In 0.537, reference position is

(a) after 0 **(b)** after 7 **(c)** after 5 **(d)** before 7

(iii) $\log_1 100$ is

(a) 2 **(b)** –2 **(c)** 0 **(d)** impossible

(iv) If $\log(x+3) = \log(15x-4)$ then x is

(a) 0.5 **(b)** 7 **(c)** 14 **(d)** 2

(v) $\log_7 7^{-3} + \log_2 4^3$ is

(a) 3 **(b)** –3 **(c)** 0 **(d)** ±3

(vi) For the log 0.00327, characteristic is

(a) –2 **(b)** –3 **(c)** 3 **(d)** 0

(vii) $\log_b(M+N)$ is

(a) $\log_b MN$ **(b)** $\log_b M + \log_b N$ **(c)** both a and b **(d)** none of these

(viii) $\log_b g^h$ is

(a) $g \log_b h$ **(b)** $\log_b(gh)$ **(c)** $(\log_b g) \times h$ **(d)** $h \log_g b$

(ix) $\log_b M - \log_b N$ is

(a) $\dfrac{\log_b M}{\log_b N}$ **(b)** $\log_b \dfrac{M}{N}$ **(c)** $\log_N M$ **(d)** $\dfrac{\log_b N}{\log_b M}$

(x) $\log_{\sqrt{10}} 100^2$ is

(a) 2 **(b)** 1 **(c)** 4 **(d)** 8

(xi) log18 is

(a) 3 log2 + log3 **(b)** log 2 + 2log3 **(c)** 3log 3 + 2log2 **(d)** 2 log 3 + 3 log 2

(xii) log 5 – log 8 + log 3 – log 2 is

(a) $\log \dfrac{5 \times 2}{8 \times 3}$ **(b)** $\log \dfrac{15}{16}$ **(c)** $\log \dfrac{30}{8}$ **(d)** log –2

(xiii) $\log_{10} 100^0$ is

(a) 2 **(b)** 0 **(c)** 1 **(d)** impossible

(xiv) Scientific notation of 6.25 is

(a) 6.25×10^1 (b) 6.25×10^0 (c) 6.25×10 (d) 0.625×10^2

(xv) Base of natural logarithm is

(a) rational number (b) integer (c) irrational number (d) 10

(xvi) If $\log_{\sqrt{x}} 25 = 4$ then x is

(a) +5 (b) –5 (c) ± 5 (d) impossible

(xvii) $\log_{\sqrt{b}} 10^4 \div \log_{\sqrt{b}} 10$ is

(a) $\log_{\sqrt{b}} \dfrac{10^4}{10}$ (b) $\log_{\sqrt{b}} 10^4 - \log_{\sqrt{b}} 10$ (c) 4 (d) $\log_{\sqrt{b}} (10^4 - 10)$

(xviii) $5 \log 2 - 2 \log 5$ is

(a) $\dfrac{(\log 2)^5}{(\log 5)^2}$ (b) $\dfrac{\log 2^5}{\log 5^2}$ (c) $\log \dfrac{2^5}{5^2}$ (d) $\dfrac{5}{2} \log \dfrac{2}{5}$

2. Convert the following into scientific notation.

(i) 53.36 (ii) 0.0000000000000102 (iii) 523.4×10^{-3}

3. Convert the following into standard notation.

(i) 7.232×10^{-2} (ii) $10.53 \times 10^2 \times 20.31$ (iii) 5.6×10^0

4. Evaluate the following.

(i) $\log_5 5^3 - \log_2 2^3$ (ii) $\log_2 4 - \log_3 1$ (iii) $\log_8(\log_x x - \log_b b^{-7})$

5. Find x if

(i) $\log_3 9 - \log_b 1 = x$ (ii) $\log_2 x - \log_2 16^{1/4} = 3$

(iii) $\log_2 (x^2 - 1) = \log_2 3$ (iv) $\log_7 x = \log_7 (8 \log_y y)$

6. If $\log_b 2 = 0.3010, \log_b 3 = 0.4771$ & $\log_b 5 = 0.6990$, then evaluate the following by applying laws of logarithms.

(i) $\log_b 30$ (ii) $\log_b 0.24$ (iii) $\log_b 360$

7. Simplify with the help of laws of logarithm $\left(\dfrac{0.5327 \times \sqrt[3]{42.97}}{0.0059}\right)^3$.

8. Prove that:

$$\log_v U \times \log_w V \times \log_u W = 1$$

UNIT
03

SETS AND RELATIONS

In this unit the students will be able to:

- Describe mathematics as a study of patterns, structures and their relationships.
- Identify sets and apply operations on three sets (subset, overlapping and disjoint cases) using Venn diagrams.
- Solve problems on classification and cataloging by using Venn diagram for scenarios involving two sets and three sets.
- Verify and apply laws of union and intersection of three sets through analytical and Venn diagram method.
- Apply concepts from set theory to real world problems (such as in demographic classification, categorizing products in shopping malls and music playlist).
- Explain product, binary relations and identify domain and range of binary relations.
- Recognize that a relation can be represented by tables, ordered pairs and graphs.

ALLAH سبحانہ وتعالی has created this huge universe and a part of it is exposed to humans but a big part of it is still unknown to humans. It is estimated that there are 200 billions to 2 trillion galaxies in the observable universe. The adjoining figure shows a set of 10 naked eye galaxies.

SET

Definition:

A set is a well-defined collection of distinct objects.

The term well-defined, means that the objects follow a given discipline with which presence or absence of some object in the set is checked.

For instance, if we say that we have a collection of lighter stones, then this collection is not well defined. Instead of this, if we say that we have a collection of stones weighing less than 1kg, this collection is well defined.

A set may consist of objects of different types.

e.g. A set of luminous objects may contain a star, a moon, a tube light or a candle.

Similarly, a set of objects present in a library may include books, tables, chairs, newspapers, keys, locks, stock registers etc.

Example 1:

(a) A set of Prime numbers which are also even i.e. {2}

(b) A set of Pakistani currency i.e.
{5, 10, 20, 50, 100, 500, 1000, 5000}

(c) A set of flowers in my garden. i.e.
{Pansy, Lilly, Daisy, Jasmine, Tulip, Rose, Hibiscus}

(d) A set of Natural numbers less than 1 i.e. { }

(e) A set of all letters present in the word 'set' {s, e, t}

(f) A set of Natural numbers among 3, – 9, 2, 3, 4, 2 i.e. {2, 3, 4}.

The repeating elements are taken once only, since repetition is not allowed in a set.

> **Key Fact**
> A set is like a box with some stuff in it which is well defined. When you look inside the box, you should be able to tell if something's in it or not.

Mathematics as a Study of Patterns, Structures and Relationships

In mathematics, patterns are more than a beautiful design. Patterns follow a predictable rule and that rule allows us to predict what will come next.

For example:

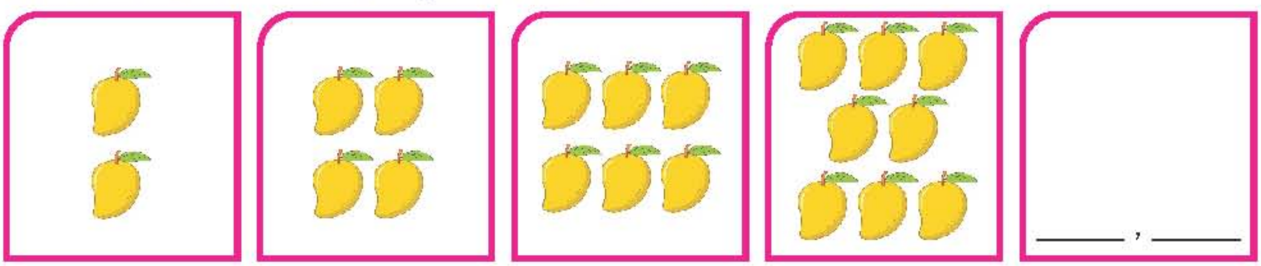
In the set of even numbers,

{2, 4, 6, 8, ….}

a pattern exists and one can determine the next number(s) in the set.

If we want to relate the above pattern with some real situation, we can ask the following question to the students:

What will be the number of mangoes in next two boxes?

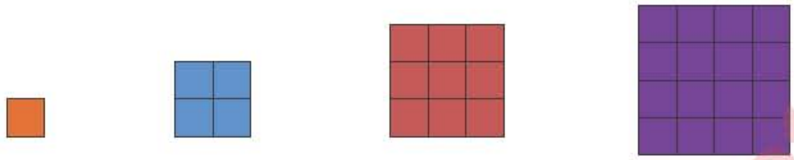


Students can easily predict the number of mangoes in the next two boxes.
Obviously, they will say:

Number of box	Number of mangoes
5	10
6	12

In the same way, we can relate the set of square of natural numbers with structural geometry as:

$$\{1^2, 2^2, 3^2, 4^2, \ldots\} = \{1, 4, 9, 16, \ldots\}$$



$1^2 = 1$ $\quad 2^2 = 4$ $\quad 3^2 = 9$ $\quad 4^2 = 16$

This example relates the patterns in numbers and geometry in the best way where the square numbers represent area of various geometrical shapes.

Key Fact

Mathematicians say that mathematics is the study of patterns in numbers and structure in geometry, and their relationships.

Check Point

Search number patterns in the set of first 100 natural numbers and relate patterns with some kind of geometrical shape or represent pattern in pictorial form.

Set Builder Form (Rule Method)

In set builder form, all the elements of a set are not listed, however we write the set by its defining rule. While writing a set in this method, some variable say x is chosen which represents all the elements of that set according to the defining rule.

e.g. A = Set of all integers, can be written in set builder form as

$A = \{x \mid x \in Z\}$ and read as

"A is the set of all elements x such that x belongs to Z"

Example 2: Write the following sets in the set builder form.

(i) B = Set of Prime numbers less than 17.

$B = \{x \mid x \in P \wedge x < 17\}$.

(ii) C = Set of multiples of 4 greater than or equal to 40.

$C = \{4x \mid x \in N \wedge x \geq 10\}$.

(iii) D = {1, 2, 3, 6}.
D = {x | x is a factor of 6}.
(iv) E = Set of squares of 1st three natural multiples of 10.
E = $\{(10x)^2 \mid x \in N \wedge 1 \le x \le 3\}$.

Example 3:

(i) The set $G = \{15x \mid x \in Z \wedge x \ge 1\}$, in descriptive form is written as
G = Set of integral multiples of 15, greater than or equal to 15.
(ii) The set $H = \{y \mid y \in W \wedge y^3 - 1 = 7\}$, in tabular form is H = {2}.
(iii) The set $I = \{x \mid x \in W \wedge -3 > x > -5\}$, in tabular form is I = { }.
(iv) The set $J = \left\{\frac{x}{2} \mid x \in E \wedge x > 0\right\}$, in tabular form is J = {1, 2, 3, 4, …}.
(v) The set $K = \{y \mid y \in Z^- \wedge y^2 = 9\}$, in descriptive method is
K = Set of negative integers containing '– 3' only.

Most Commonly used Sets of Numbers

i. N = Set of Natural numbers = {1, 2, 3, 4,……..}
ii. W = Set of Whole numbers = {0, 1, 2, 3, 4,…….}
iii. Z = Set of Integers = {0, ± 1, ± 2, ± 3,....}
iv. E = Set of Even numbers = {0, ± 2, ± 4, ± 6,….}
v. O = Set of Odd numbers = {± 1, ± 3, ± 5, ± ,…}
vi. P = Set of Prime numbers = {2, 3, 5, 7, 11, 13,…..}
vii. Q = Set of Rational numbers $= \left\{\frac{p}{q} / p, q \in Z \wedge q \ne 0\right\}$
viii. Q' = Set of Irrational numbers $= \left\{x / x \ne \frac{p}{q}, p, q \in Z \wedge q \ne 0\right\}$
ix. R = Set of Real numbers $= \{x / x \in Q \vee x \in Q'\}$

Check Point
Can we write set of Real numbers in tabular form? Justify!

The above mentioned sets from (i-vi) are first written in descriptive method and then in tabular form (Roster method).
However, the sets from (vii-ix) are first written in descriptive method then in set builder form.

Set Operations

Union of Sets

Let A and B be two given sets. Then union of A and B is the set of all those elements which are taken either from A or from B or from both.

The union of A and B is denoted by '$A \cup B$' and:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

e.g. If A = {1, 2, 3, 4, 5} and B = {3, 4, 1, 7, 6}, then:

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

Intersection of Sets

Let A and B be given sets then the intersection of A and B is the set of elements which belong to both A and B.

The intersection of A and B is denoted by '$A \cap B$' and:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

e.g. If A = {1, 2, 3, 4, 5} and B = {3, 4, 1, 7, 6}, then:

$$A \cap B = \{1, 3, 4\}$$

Difference of Sets

Difference of two sets A and B, denoted by A – B is a collection of those elements of A which are not present in B. For the two sets A and B:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Let A = {1, 2, 3, 4, 5} and B = {3, 4, 6, 7, 8}, then:

A – B = {1, 2, 3, 4, 5} – {3, 4, 6, 7, 8} = {1, 2, 5}

In general, $A - B \neq B - A$

Complement of a Set

Let $A \subset U$ (i.e. A is proper subset of the universal set). Then the set of all elements of U, which are not in A, is called complement of A.

Complement of A is denoted by A^c or $A' = U - A$ and is defined as:

$$\{x \mid x \in U \wedge x \notin A\}$$

e.g. If U = {1, 2, 3, 4, 5, 6, 7}, A = {2, 4, 6}, then:

$$A^c = U - A = \{1, 3, 5, 7\}$$

Key Fact

- The common elements of A and B are written once.
- Two sets A and B are said to be disjoint, if $A \cap B = \phi$
- The elements of A are never present in A^c and vice versa.
- Two sets are said to be overlapping, if neither set is subset of the other and their intersection is non-empty.

Union and Intersection of Three Sets

Disjoint Sets

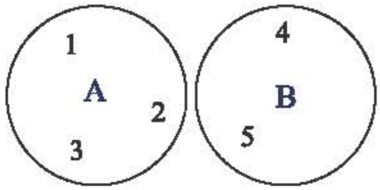
Two sets A and B are said to be disjoint if they have no common elements. i.e. $A \cap B = \phi$.

For example,

A = {1, 2, 3} and B = {4, 5} are disjoint sets as both sets have no common element. i.e.

$$A \cap B = \{1, 2, 3\} \cap \{4, 5\} = \phi$$

Venn diagram representing above disjoint sets is:



Overlapping Sets

Two sets A and B are called overlapping if:

(i) There is at least one element common in both the sets.

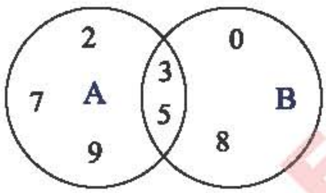
(ii) Neither of the sets is a subset of other set.

For example, the sets

$A = \{2, 3, 5, 7, 9\}$ and $B = \{0, 3, 5, 8\}$ are overlapping as

$A \cap B = \{3, 5\} \neq \phi$ and A and B are not subsets of each other.

Venn diagram representing above overlapping sets is:



Key Fact

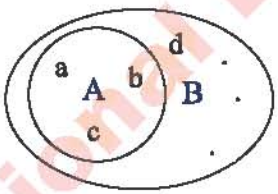
Venn diagrams are used to explain the whole set theory in a very simple way.

Subset

Set A is called subset of a set B if every element of set A is also an element of B.

For example, the set $A = \{a, b, c\}$ is a subset of $B = \{a, b, c, d, \ldots\}$ as all elements of set A are also elements of set B.

Venn diagram representing above subset case is:



The pictorial representation of the relationship $A \subset U$ is called a Venn diagram. In a Venn diagram, universal set is represented by the interior of the rectangle, however inside the rectangle, the subsets are represented by the interior of any other closed shape like circles or ovals etc.

Union and Intersection of Three Sets using Venn Diagrams

To find the union and intersection of three sets say $A \cup B \cap C$, first we find $A \cup B$ or $B \cap C$ and then find the union or intersection with remaining set.

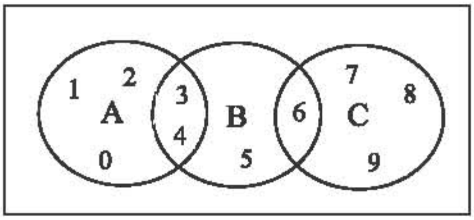
We use brackets to separate two sets from the third one because these represent different sets.

i.e. $(A \cup B) \cap C$ or $A \cup (B \cap C)$

Example 4:

Represent sets $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{6, 7, 8, 9\}$ through Venn diagram.

Solution:

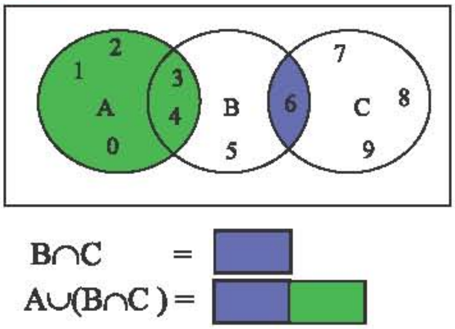


Example 5:

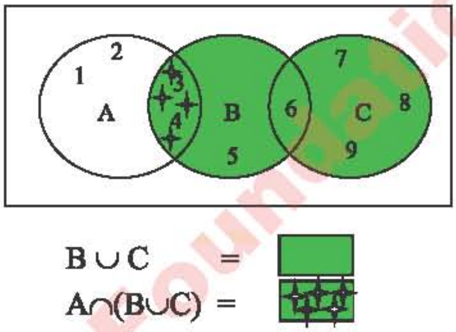
If A = {0, 1, 2, 3, 4}, B = {3, 4, 5, 6} and C = {6, 7, 8, 9}, find $A \cup (B \cap C)$ and $A \cap (B \cup C)$ through Venn diagram.

Solution:

(i)



(ii)



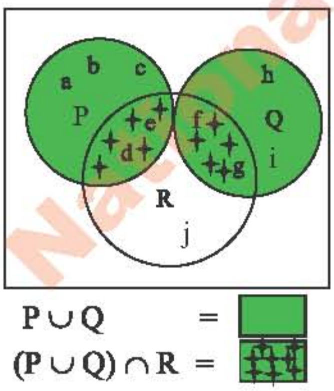
Example 6:

If P = {a, b, c, d, e}, Q = {f, g, h, i}, R = {d, e, f, g, j}

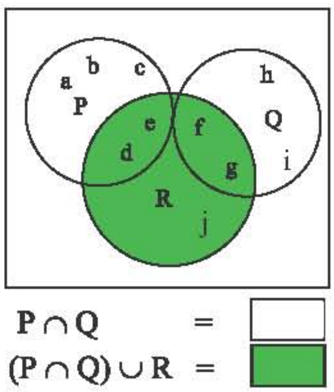
Find $(P \cup Q) \cap R$ and $(P \cap Q) \cup R$ through Venn diagram.

Solution:

(i)

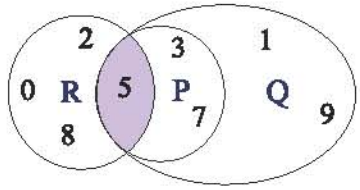


(ii)

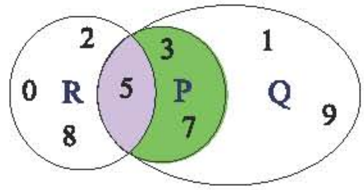


Example 7: If P = {3, 5, 7}, Q = {1, 3, 5, 7, 9}, R = {0, 2, 5, 8}, then find $P \cup (Q \cap R)$ using Venn diagram.

Solution: P = {3, 5, 7}, Q = {1, 3, 5, 7, 9}, R = {0, 2, 5, 8}



$Q \cap R = \{5\} =$ ▭



$P \cup (Q \cap R) = \{3, 5, 7\} =$ ▭▭

Verification of Associative Laws Using Venn Diagram

We illustrate the concept with the help of following examples.

Associative Property of Union

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof:

Let $x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\Rightarrow$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ (a)

Similarly, we can prove that:

$A \cup B \cup C) \subseteq (A \cup B) \cup C$ (b)

From (a) and (b), we have:

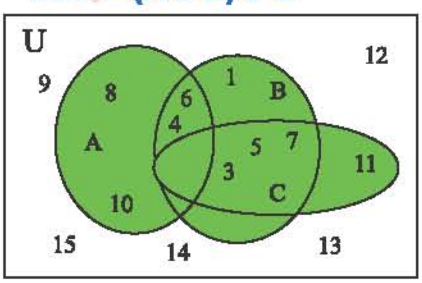
$$(A \cup B) \cup C = A \cup (B \cup C)$$

Example 8:

If $U = \{1, 2, 3, 4, 5, \ldots, 15\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$ and $C = \{2, 3, 5, 7, 11\}$, then verify the associative property of union using Venn diagram.

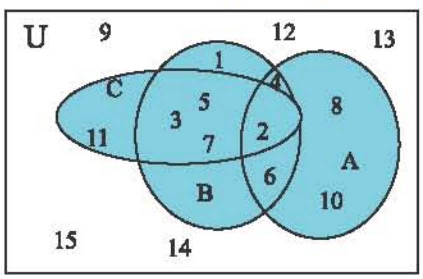
Solution:

LHS = $(A \cup B) \cup C$



$(A \cup B) \cup C =$ ▭

RHS = $A \cup (B \cup C)$



$A \cup (B \cup C) =$ ▭

From both the figures, it is observed that same region is shaded.

i.e. $(A \cup B) \cup C = A \cup (B \cup C)$

$\therefore$ Associative property of Union is verified.

Associative Property of Intersection

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Proof:

Let $y \in (A \cap B) \cap C$

$\Rightarrow$ $y \in A \cap B$ and $y \in C$

$\Rightarrow$ $(y \in A$ and $y \in B)$ and $y \in C$

$\Rightarrow$ $y \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow$ $y \in A$ and $y \in (B \cap C)$

$\Rightarrow$ $y \in A \cap (B \cap C)$

$\Rightarrow$ $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ (a)

Similarly, we can prove that:

$A \cap (B \cap C) \subseteq (A \cap B) \cap C$ (b)

From (a) and (b), we have:

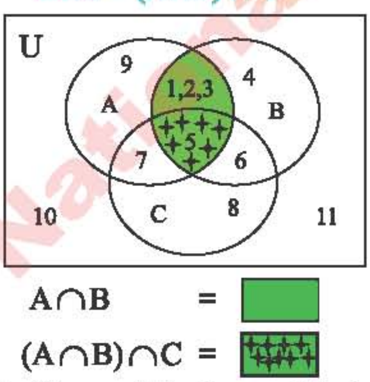
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Example 9:

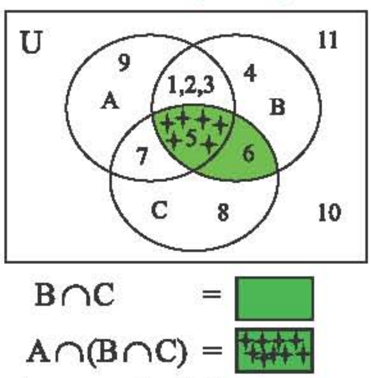
If $U = \{1, 2, 3, \ldots, 11\}$, $A = \{1, 2, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and $C = \{5, 6, 7, 8\}$ then verify the associative property of intersection.

Solution:

LHS = $(A \cap B) \cap C$



RHS = $A \cap (B \cap C)$



From both the figures it is observed that same regions are shaded.

i.e. $(A \cap B) \cap C = A \cap (B \cap C)$

$\therefore$ Associative property of intersection is verified.

Verification of Distributive Laws Using Venn Diagram

(a) Distributive Property of Union over Intersection

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (a)

Again, let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ or $y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (b)

From (a) and (b), we have:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(b) Distributive Property of Intersection over Union

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof:

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ (a)

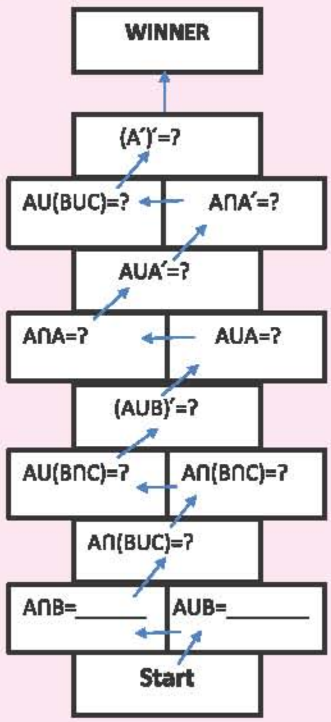
Again, let $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$

Math Play Ground

1. Take students to the playground and make a hopscotch as shown:
2. Ask a student to start hoping and filling the blanks.



$\Rightarrow \quad y \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow \quad y \in A$ and $y \in (B \cup C)$

$\Rightarrow \quad y \in A \cap (B \cup C)$

$\Rightarrow \quad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \qquad$ (b)

From (a) and (b), we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 10: Verify through Venn diagram

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

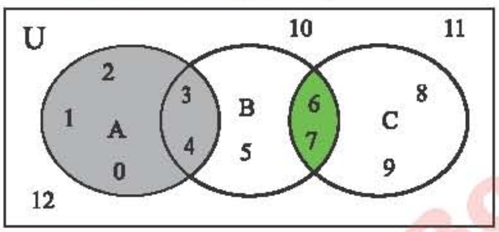
When $U = \{x : x \in W \wedge x \leq 12\}$, $A = \{x : x \in W \wedge x \leq 4\}$

$B = \{y : y \in N \wedge 3 \leq y \leq 7\}$ and $C = \{z : z \in N \wedge 3 \leq z \leq 7\}$

Solution:

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

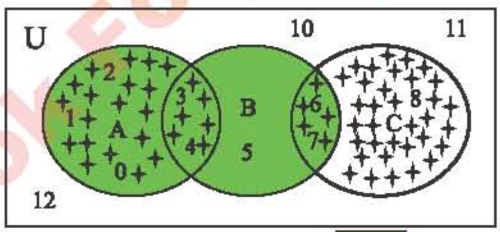
LHS = $A \cup (B \cap C)$



$B \cap C$ =

$A \cup (B \cap C)$ =

RHS = $(A \cup B) \cap (A \cup C)$



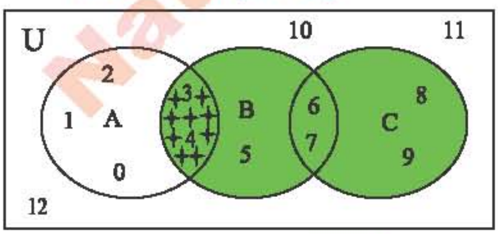
$A \cup B$ =

$A \cup C$ =

$(A \cup B) \cap (A \cup C)$ =

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C$

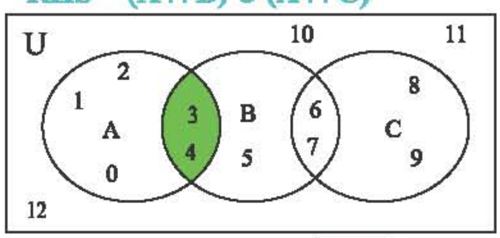
LHS = $A \cap (B \cup C)$



$B \cup C$ =

$A \cap (B \cup C)$ =

RHS = $(A \cap B) \cup (A \cap C)$



$A \cap B$ =

$A \cap C$ =

$(A \cap B) \cup (A \cap C)$ =

Key Fact

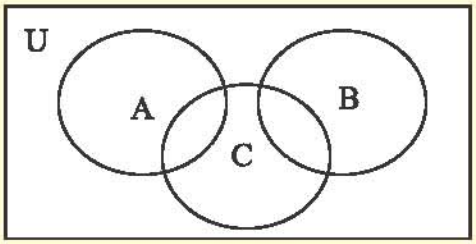
(a) If $A = \{0, 1, 3, 5, 7\}$, $B = \{x \mid x \in N \wedge x \leq 5\}$ and $C = \{1, 2, 3, 4, 6, 12\}$, then find $(A \cup B) \cap C$ using Venn diagrams.

(b) If $A = \{0, 2, 4, 6\}$, $B = \{1, 3, 5, 7\}$ and $C = \{1, 2, 3, 6\}$, then find $(A \cap B) \cap C$, $A \cap (B \cap C)$ and $(A \cup B) \cap C$ using Venn diagram.

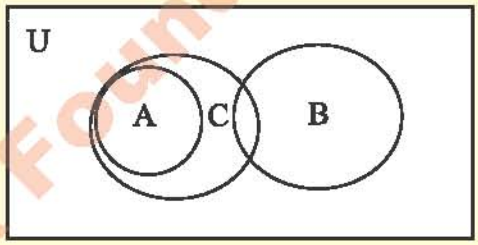
EXERCISE 3.1

1. Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$, $(A \cup B) \cup C$ and $A \cap (B \cap C)$ using following Venn diagrams.

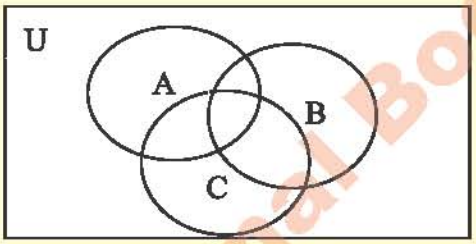
(i)



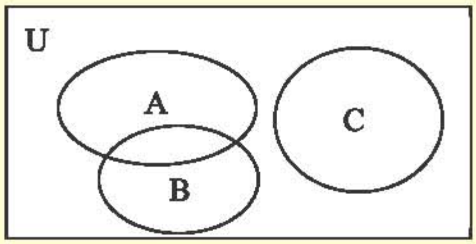
(ii)



(iii)



(iv)



2. If $X = \{a, b, c, d, e\}$, $Y = \{a, c, e\}$, $Z = \{g, h, i, j\}$ then, find the following using Venn diagram.

   (i) $(X \cup Y) \cup Z$ (ii) $X \cup (Y \cup Z)$ (iii) $(X \cap Y) \cap Z$
   (iv) $X \cap (Y \cap Z)$ (v) $(X \cup Y) \cap Z$ (vi) $(X \cap Y) \cup Z$

3. Verify associative law of union and intersection by using diagrams of question 1.

4. Verify:
   (i) distributive property of union over intersection,
   (ii) distributive property of intersection over union,
   by using diagrams of question 1.

5. Prove by using Venn diagram:

   (a) $(P \cup Q) \cup R = P \cup (Q \cup R)$ (b) $(P \cap Q) \cap R = P \cap (Q \cap R)$

   when (i) $P = \{0, 1, 2, 3\}$, $Q = \{2, 3, 4, 5, 6\}$, $R = \{5, 6, 7, 8, 9\}$

(ii) $P = \{m, n, o, p, q\}$, $Q = \{r, s, t, u\}$, $R = \{t, u, v, w\}$

6. Verify $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ using Venn diagram for the following sets.
$X = \{-1, -2, -3\}$, $Y = \{0, 1, 2, 3\}$, $Z = \{0, \pm 1, \pm 2, \pm 3\}$

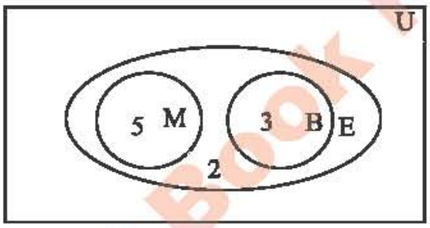
7. Verify $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
X = Set of first three Vowels, Y = Set of letters of the word " energy",
Z = Set of letters of the word "algebra"

Application of Venn Diagrams

Venn diagram is a practical mathematical tool for solving real world problems of set theory. Few of the examples depict the vital role of the Venn diagrams in problem solving.

Example 11: Among the ten teachers of a secondary school, five teach Mathematics and three teach Biology. However, all these teachers also teach English? Show the data by Venn diagram. Also find how many teachers teach only English.

Solution: If E represents set of English Teachers, M represents set of Mathematics Teachers, B represents the set of Biology Teachers and then both M and B are the subsets of E, as shown in the figure below.

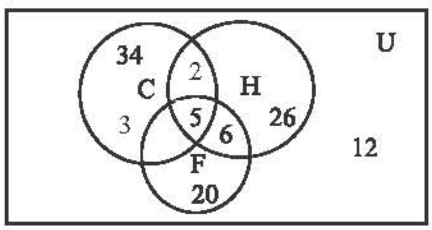


From the Venn diagram it is observed that two of the teachers only teach English.

Example 12: In a survey, people were asked whether they like cricket, hockey or football. Using the Venn diagram find the number of people playing:

(i) only cricket,
(ii) hockey and football,
(iii) all three games,
(iv) either of the three games,
(v) neither of the three games,

Also find number of people surveyed.



Solution: From Venn diagram, it is clear that:

(i) Number of people who play only cricket = 34
Which shows $n(C - C \cap H \cap F)$

(ii) Number of people who play both hockey and football = 6 + 5 = 11
Which shows $n(H \cap F)$

(iii) Number of people who play all three games = 5
Which shows $n(C \cap H \cap F)$

(iv) Number of people who play either of the games = 34 + 5 + 6 + 26 + 20 = 91
Which shows $n(C \cup H \cup F)$

(v) Number of people who do not play any game = 12
Which shows $n(U - C \cup H \cup F)$

Total number of people = n(U) = 34 + 5 + 6 + 26 + 20 + 12 = 103

Application of Set Theory

Applications of set theory are most frequently used in science and mathematics fields like biology, chemistry, and physics as well as in computer and electrical engineering. These applications range from forming logical foundations for all branches of mathematics. Therefore, understanding set theory is crucial for learning many subjects.

Following formulae are helpful in the set theory.

(i) For any two overlapping sets A and B:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A \cup B) - n(B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

Key Fact

For any sets A and B:

$A \cup A = A,\quad A \cap A = A,\quad A \cap \emptyset = \emptyset$

$A \cup \emptyset = A,\quad A \cap B \subseteq A,\quad A \subseteq A \cup B$

(ii) For any two sets A and B that are disjoint:

- $n(A \cup B) = n(A) + n(B)$
- $n(A - B) = n(A)$

(iii) For any three sets A, B and C:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Example 13:

In a class of 80 students, 40 like English, 34 like Mathematics and 9 like both. How many students like either of both subjects and how many like neither?

Solution:

Total number of students = n(T) = 80

Number of students that like English = n(E) = 40

Number of students that like Mathematics = n(M) = 34

Therefore, total number of students that like both subjects is:

$$n(E \cup M) = n(E) + n(M) - n(E \cap M)$$
$$= 40 + 34 - 9 = 65$$

Number of students that do not like both subjects is:

$$n(T) - n(E \cup M) = 80 - 65 = 15$$

EXERCISE 3.2

1 Let A and B be two finite sets such that $n(A) = 24$, $n(B) = 18$ and $n(A \cup B) = 31$. Find $n(A \cap B)$.

2. If $n(A - B) = 23$, $n(A \cup B) = 44$ and $n(A \cap B) = 2$, then find $n(B - A)$. Also find $n(B)$. (Hint: $n(B) = n(A \cap B) + n(B - A)$)

3. In a group of 30 Mathematics students, 20 like Algebra and 15 like both Geometry and Algebra. Show the data by Venn diagram. Also find how many students like Geometry.

4. In a street with 50 houses, 25 houses have lawns, 32 houses have car porch and 15 houses have both lawn and car porch. Show the data by Venn diagram. Also find how many houses have neither lawn nor porch.

5. In a survey of 940 children, 400 students were found studying at primary level, 240 students at elementary and 175 at secondary level. Create a Venn diagram to illustrate this information. How many children were found out of school?

6. ABC Dairy polls its customers on their favorite flavor: chocolate, vanilla or mango? 100 customers said they like mango flavor, 90 customers said they like vanilla, 40 polled for chocolate, 20 customers liked both mango and vanilla while 14 liked both chocolate and vanilla. How many customers said they like:

   (i) only mango? (ii) only vanilla (iii) only chocolate

7. In a survey of university 200 students were interviewed. It was found that:
   42 students have laptops, 80 students have cell phones, 100 students have iPods,
   23 students have both a laptop and a cell phone, 10 students have both a laptop and iPod,
   14 students have both a cell phone and iPod and 8 students have all three items.

   (a) How many students have only cell phone?
   (b) How many students have none of the three items?
   (c) How many students have both iPod and laptop but not cellphone?

8. In a girl college, every student plays either badminton or table tennis or both. If 350 students play badminton, 280 play table tennis and 150 play both. Find how many students are there in the college?

9. Among 50 students, 8 are learning both English and Chinese languages. A total of 26 students are learning English. If every student is learning at least one language, how many students are learning Chinese?

10. Out of 70 people, 48 like tea and 40 like coffee and each person likes at least one of the two drinks. How many like both tea and coffee?

11. There are 46 students in science group and 50 students in arts group. Find the number of students who are either in science or arts group.

12. In a group of people, 52 people can speak Arabic and 112 can speak French. How many can speak Arabic only? How many can speak French only if 12 of them can speak both languages? How many people were in the group?
13. In a high school, 360 students like reading story books, 170 like practical activities and 150 like both. Find
 (i) The number of students who like reading story books only.
 (ii) The number of students who like only practical activities.
 (iii) The total number of students in the school.
14. In a survey of 60 people, it was found that 25 people watch channel A, 16 watch channel B, 13 watch channel C, 4 watch both A and B, 7 watch both B and C, 8 watch both A and C, 3 watch all three channels. Find the number of people who watch at least one of the channels? Also find number of people who do not watch these channels.

Binary Relations

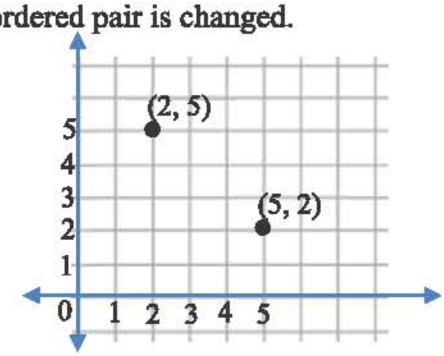
Ordered Pair

Pairs of two numbers in which order of numbers is not invertible, is called an ordered pair. The numbers in an ordered pair are written within small brackets (parenthesis) and are separated by comma.

For example, (a, b) is an ordered pair in which a is called first element and b is called second element. By interchanging the positions of elements, the ordered pair is changed.

As in geometry, position of a point is determined by ordered pair, therefore (2, 5) and (5, 2) represent two different points.

Thus, $(2, 5) \neq (5, 2)$



Equality of Two Ordered Pairs

Two ordered pairs (a, b) and (c, d) are equal if:

$$a = b \text{ and } c = d$$

Key Fact

- $(a, b) \neq (b, a)$
- $(a, b) = (c, d)$
 $\Leftrightarrow a = c \;\&\; b = d$

Example 15:

Find the values of x and y when $(x - 3y, 5x + 1) = (4, 6)$

Solution: By the equality of ordered pairs, we have:

$x - 3y = 4$ (i)

$5x + 1 = 6$ (ii)

From equation (ii), $5x = 6 - 1$

$5x = 5 \quad \Rightarrow \quad x = 1$

Substituting $x = 1$ in equation (i), we get:

$1 - 3y = 4 \Rightarrow -3y = 3$

$\Rightarrow y = -1$

Check Point

Find a and b when:
$(a + 1, 4) = (2, b - 3)$

Cartesian Product of Sets

If A and B are two non-empty sets, then Cartesian product $A \times B$ is the set of ordered pairs (x, y) such that $x \in A$ and $y \in B$. Mathematically:

$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

Similarly, $B \times A = \{(y, x) \mid y \in B \wedge x \in A\}$

e.g. If $A = \{0, 1, 2\}$, $B = \{3, 4\}$

then $A \times B = \{0, 1, 2\} \times \{3, 4\}$

$= \{(0, 3), (0, 4), (1, 3), (1, 4), (2, 3), (2, 4)\}$

Key Fact

- $n(A \times B) = n(A) \times n(B)$
- $A \times B = \phi$ if either $A = \phi$ or $B = \phi$

$A \times B$ can also be represented through table as follows:

A × B	0	1	2
3	(0, 3)	(1, 3)	(2, 3)
4	(0, 4)	(1, 4)	(2, 4)

In the same way, we can find $B \times A$ as:

$B \times A = \{3, 4\} \times \{0, 1, 2\}$

$= \{(3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2)\}$

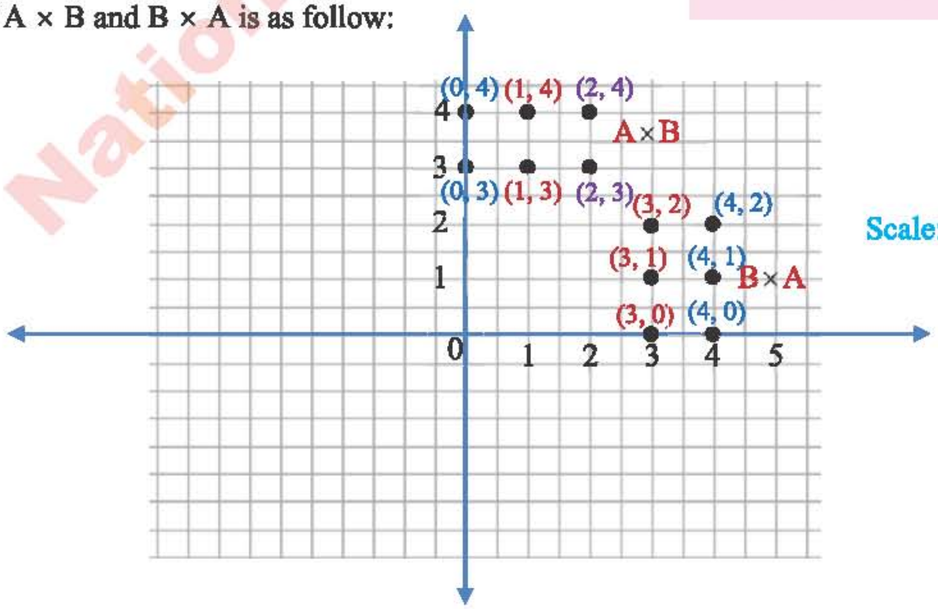
Table for $B \times A$ is:

B × A	3	4
0	(3, 0)	(4, 0)
1	(3, 1)	(4, 1)
2	(3, 2)	(4, 2)

Key Fact

- Each element of the set $A \times B$ is called an ordered pair.
- The ordered pair $(a, 1)$ cannot be written as $(1, a)$.
- The number of subsets of $A \times B = 2^{n(A \times B)}$.

Graph of $A \times B$ and $B \times A$ is as follow:



Scale: 2 squares = 1

From the definition, table and graph, we see that:

$$A \times B \neq B \times A$$

Key Fact

- In general $A \times B \neq B \times A$
- $A \times B = B \times A$ if and only if $A = B$.
- $n(A \times B) = n(B \times A)$

There are many subsets of $A \times B$

e.g. $R_1 = \phi$, $R_2 = \{(0, 3)\}$, $R_3 = \{(1, 3), (0, 4)\}, \ldots$

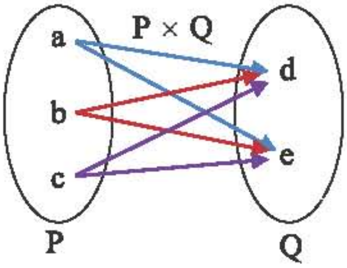
Example 16:

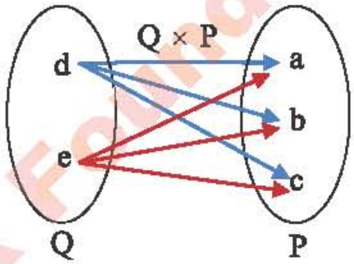
Exhibit $P \times Q$ and $Q \times P$ by arrow diagram when $P = \{a, b, c\}$ and $Q = \{d, e\}$.

Solution:

$P \times Q = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$

$Q \times P = \{(d, a), (d, b), (d, c), (e, a), (e, b), (e, c)\}$





EXERCISE 3.3

1. Find the values of unknowns when:

   (i) $(a, -b) = (7, 1)$ (ii) $(2a, 2b + 3) = (-10, -b)$ (iii) $(2a - 4, 6) = (8, -b + 1)$

   (iv) $(x + 2y, \ y - 3) = (2, 5)$ (v) $(2x - y, \ y - 3x) = (4, 2)$

   (vi) $(4x + 6y, \ x - 12y) = (6, -3)$ (vii) $(5x + y, \ -x + y) = (6, 1)$

2. Let $A = \{1, 4, 8\}$ and $B = \{1, 0\}$. Find:

   (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

   How many elements are there in $A \times B$, $B \times A$, $A \times A$ and $B \times B$?

3. Let $E = \{1, 3\}$ and $F = \{4, 6, 8\}$. Express $E \times F$, $F \times E$, $E \times E$, $F \times F$ graphically.

4. If $L \times M = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$, then find sets L, M and $M \times L$.

5. Given that $A = \{1, 3, 5\}$, $B = \{2, 4\}$, $C = \{6, 7\}$.

   (i) Find $A \times (B \cup C)$ (ii) Find $(A \times B) \cup (A \times C)$

   (iii) Verify $A \times (B \cup C) = (A \times B) \cup A \times C)$

6. Given that $D = \{a, e, i\}$, $E = \{a, c\}$, $F = \{b, c\}$.

   (i) Find $D \times (E \cap F)$ (ii) Find $(D \times E) \cap (D \times F)$

   (iii) Verify $D \times (E \cap F) = (D \times E) \cap (D \times F)$

7. Given that $A = \{x / x \in N, x < 3\}$, $B = \{y / y \in W, y < 2\}$, $C = \{0, 2, 4\}$.
   (i) Verify $A \times (B - C) = (A \times B) - (A \times C)$
   (ii) Verify $(A - B) \times C = (A \times C) - (B \times C)$
8. Let $X = \{x / x \in W, x \leq 2\}$ and $Y = \{-1, -2, -3\}$. Exhibit $X \times Y$ and $Y \times X$ by arrow diagram.

Binary Relation

A binary relation R in the set A × B is a subset of the Cartesian product A × B.
Symbolically R is the relation in a set A × B if and only if $R \subseteq A \times B$.
If R is elation from A to B, then:

$$R = \{(a, b) / a \in A, b \in B\}$$

A binary relation can also be taken from only one set after taking Cartesian product of the set with itself e.g. from A × A
If R_1 is elation from A to A, then:

$$R_1 = \{(a, b) / a \in A, b \in A\}$$

Key Fact
- A binary relation is a set of ordered pairs.
- Number of binary relations in $A \times B = 2^{n(A \times B)}$

Example 17:
If A = {5, 10, 15, 20, 25} then find the number of binary relations in A.

Solution: Number of elements in A = n (A) = 5
Number of elements in A × A = n (A × A) = 5 × 5 = 25
Number of binary relations in A (or A × A) = 2^{25}

Example 18:

If $P = \{2, 3\}$, $Q = \left\{\frac{1}{2}, \frac{1}{3}\right\}$, then find all possible binary relations in P×Q.

Solution:
Number of elements in P = n (P) = 2
Number of elements in Q = n (Q) = 2
Number of elements in P × Q = n (P×Q) = n(P) × n(Q)
$= 2 \times 2 = 4$
Number of binary relations in P × Q = $2^{n(P \times Q)}$
$= 2^{2\times2} = 2^4 = 16$

Now $P \times Q = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$. Here,

$R_1 = \phi$, $R_2 = \left\{\left(2, \frac{1}{2}\right)\right\}$, $R_3 = \left\{\left(2, \frac{1}{3}\right)\right\}$, $R_4 = \left\{\left(3, \frac{1}{2}\right)\right\}$, $R_5 = \left\{\left(3, \frac{1}{3}\right)\right\}$,

$R_6 = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right)\right\}$, $R_7 = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{2}\right)\right\}$, $R_8 = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$,

$R_9 = \left\{\left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right)\right\}$, $R_{10} = \left\{\left(2, \frac{1}{3}\right), \left(3, \frac{1}{3}\right)\right\}$, $R_{11} = \left\{\left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$,

$R_{12} = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right)\right\}$, $R_{13} = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{3}\right)\right\}$

$R_{14} = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$, $R_{15} = \left\{\left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$

$R_{16} = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$

Example 19:

Let A = {1, 2, 3}, B ={0, 1, 3} and R = {(a, b) / $a \in$ A, $b \in$ B} and $a > b$}, then find R and show it by arrow diagram.

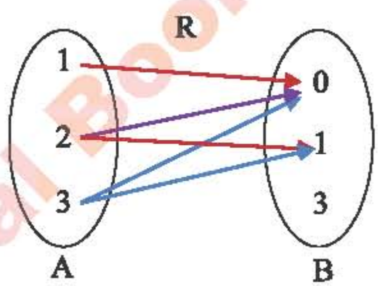
Solution:

A = {1, 2, 3}, B ={0, 1, 3}

A × B = {(1, 0), (1, 1), (1, 3), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1), (3, 3)}

Now R = {(a, b) / $a \in$ A, $b \in$ B} and $a > b$}

R = {(1, 0), (2, 0), (2, 1), (3, 0), (3, 1)}



Domain of Binary Relation

Set of all the first elements of ordered pairs in a binary relation is called domain of that binary relation.

In example 18, Domain of R_{11} = {3} and Domain of R_{14} = {2, 3}.

Range of Binary Relation

Set of all the second elements of ordered pairs in a binary relation is called range of that binary relation.

In example 18, Range of $R_6 = \left\{\frac{1}{2}, \frac{1}{3}\right\}$ and Range of $R_7 = \left\{\frac{1}{2}\right\}$.

Example 20:

(a) If A = Set of Natural numbers and $R = \{(x, y) \mid x \in A \wedge y \in A\}$ i.e. $R \subseteq A \times A$

Then find the domain and range of R.

(b) If $T = \{0, \pm 1, \pm 2\}$ and $R_1 = \{(x, y) \mid x \in T \wedge y \in T \wedge x + y = 0\}$, then find the Dom R and Range R.

(c) If $E = \{2, 4, 6\}$, $F = \{0, 1, 2\}$ and $R_2 = \{(x, y) \mid x \in E, y \in F \wedge x + y = 6\}$, then

(i) Write $E \times F$ **(ii)** Write R_2 in tabular form **(iii)** Find Dom R_2 and Range R_2.

Solution:

(a) $R = \{(1, 1), (1, 2), (1, 3), \ldots, (2, 1), (2, 2), (2, 3), \ldots (3, 1), (3, 2), (3, 3), \ldots\}$

So, Domain of $R = \{1, 2, 3, \ldots\}$ = Set of Natural numbers

Range of $R = \{1, 2, 3, \ldots\}$ = Set of Natural numbers

$\therefore$ Dom R = Range R = A

(b) $R_1 = \{(0, 0), (-1, 1), (1, -1), (-2, 2), (2, -2)\}$

Dom $R_1 = \{0, \pm 1, \pm 2\}$, Range $R_1 = \{0, \pm 1, \pm 2\}$

(c) $E \times F = \{(2, 0), (2, 1), (2, 2), (4, 0), (4, 1), (4, 2), (6, 0), (6, 1), (6, 2)\}$

$R_2 = \{(4, 2), (6, 0)\}$

Dom $R_2 = \{4, 6\}$, Range $R_2 = \{0, 2\}$

Key Fact

$R_1 = \phi$ is called a void relation.

Inverse Relation

Let $R = \{(a, b) / a \in A, b \in B\}$ be a relation from A to B then the inverse of R is defined by:

$R^{-1} = \{(b, a) / b \in B, a \in A\}$

Example 21:

Given that $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 9, 16\}$ and $R = \{(x, y) \mid x \in A \wedge y \in B \wedge x^2 = y\}$.

Verify: Dom R^{-1} = Range R and Range R^{-1} = Dom R

Solution:

Given $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 9, 16\}$ and $R = \{(x, y) / x \in A, y \in B\} \wedge x^2 = y\}$

R in tabular form is:

$R = \{(0, 0), (2, 4), (3, 9)\}$

Dom $R = \{0, 2, 3\}$ and Range $R = \{0, 4, 9\}$

Inverse of R is:

$R^{-1} = \{(0, 0), (4, 2), (9, 3)\}$

Dom $R^{-1} = \{0, 4, 9\}$ and Range $R^{-1} = \{0, 2, 3\}$

Which shows that:

Dom R^{-1} $\{0, 4, 9\}$ = Range R and Range $R^{-1} = \{0, 2, 3\}$ = Dom R

Key Fact

- Dom R^{-1} = Range R
- Range R^{-1} = Dom R

EXERCISE 3.4

1. Find the number of binary relations in the following cases.
   (i) $A = \{1, 3\}, B = \{0, 2, 4\}$ (ii) $n(C) = 7$ (iii) $D = \{1, 3, 5\}$
2. Find all possible binary relations in the following cases mentioning the number of binary relations in each case.
   (i) $A = \left\{\sqrt{2}, \sqrt{3}, \sqrt{5}\right\}$, $B = \left\{\sqrt[3]{5}\right\}$ (ii) $C = \{\pi, e\}$ (iii) $D = \{5\}, E = \{1, 10\}$
3. If $P = \{7, 8, 9\}$ then find 2 binary relations from P to P. Also find domain and range of each relation.
4. Let $H = \{5, 6, 7, 8, 9\}$ and $G = \{5, 7, 9, 11\}$. Write the following relations from H to G in tabular form.
   (i) 'is equal to' (ii) 'is less than' (iii) 'is greater than'
   (iv) 'is one less than' (v) 'is one greater than' (vi) 'is two less than'
5. Let $C = \{2, 4, 6\}$, $D = \{4, 6, 8, 9, 12\}$ and $R = \{(x, y) / x \in C, y \in D\} \wedge x$ is factor of $y\}$.
   (i) Write R in tabular form. (ii) Find domain and range of R.
   (iii) Find R^{-1}. (iv) Represent R by arrow diagram.
6. Let $R = \{(2, 0), (4, 2), (6, 4), (8, 6), (10, 8)\}$
   (i) Write R in set builder form. (ii) Find domain and range of R.
   (iii) Write R^{-1} in tabular and set builder from.
   (iv) Represent R and R^{-1} by arrow diagram.
7. Let $A = \{0, 1, 3\}$ and $B = \{1, 2, 3, 5, 7\}$. Write $R = \{(x, y) \mid x \in A, y \in B \wedge y = 2x + 1\}$ in tabular form. Also find R^{-1}.
8. If $S = \{1, 2, 4, 8\}$, $T = \{3^0, 3^1, 3^2\}$, then write the following binary relations in tabular form.
9. Find the domain and range in each case.
   (i) $R_1 = \{(x, y) \mid x \in S, y \in T \wedge x = y\}$ (ii) $R_2 = \{(x, y) \mid x \in S, y \in T \wedge y < x\}$
   (iii) $R_3 = \{(x, y) \mid x \in S, y \in T \wedge x + y \in E\}$ (iv) $R_4 = \{(x, y) \mid x \in S, y \in T \wedge x \times y \in O\}$
   (v) $R_5 = \{(x, y) \mid x \in S, y \in T \wedge y > 2x\}$

KEY POINTS

- If A and B are any two sets, then the set consisting of all the elements of these two sets is called union of these two sets.
- If A and B are any two sets then the set consisting of all the common elements of these two sets is called intersection of these two sets.
- The pictorial representation of any set is called a Venn diagram.
- We can prove associative laws and distributive laws by using Venn diagrams.
- Any subset of the Cartesian product $A \times B$ is a binary relation.

MISCELLANEOUS EXERCISE 3

1. Encircle the correct option.

i. Set builder form of A – B is:

(a) $\{x \mid x \in A\}$ (b) $\{x \mid x \in A \wedge x \notin B\}$ (c) $\{x \mid x \in A \wedge x \in B\}$ (d) $\{x \mid x \in B\}$

ii. If $A \cup B = A$ and $A \cap B = B$, then:

(a) $A \subset B$ (b) $A \not\supseteq B$ (c) $A \supseteq B$ (d) $A \neq B$

iii. If $A \subset B$ then $A - B =$

(a) A (b) B (c) ϕ (d) B – A

iv. If $A - B = B - A = \phi$, then:

(a) A = B (b) $B \subseteq A$ (c) $A \subseteq B$ (d) all a, b & c

v. Set of common elements of A and A^c is _____ set.

(a) infinite (b) null (c) universal (d) singleton

vi. Set of real numbers can be written in:

(a) tabular form. (b) descriptive form.
(c) set builder form. (d) both b and c.

vii. If R = {(2, 1), (4, 3), (2, 2)}, then Dom R =

(a) {2, 4, 2} (b) {2, 4} (c) {1, 3, 2} (d) none of these

viii. If $R = \{(a, b) / a, b \in Z \wedge a + b = 0\}$, then:

(a) Dom R $\subset$ Range R (b) Range R $\subset$ Dom R
(c) Dom R = Range R (d) none of these.

ix. If $R = \{(a, b) / a, b \in N \wedge a \times b = 12\}$ then tabular form of R is:

(a) {(1, 12), (2, 6), (3, 4), (4, 3), (6, 2), (12, 1)} (b) {(1, 12), (2, 6), (3, 4)}
(c) {(12, 1), (6, 2), (4, 3)} (d) {(1, 12), (4, 3), (2, 6)}

x. If n (A) = p, then n $(A \times A)$ =

(a) p (b) 2p (c) $2p^2$ (d) p^2

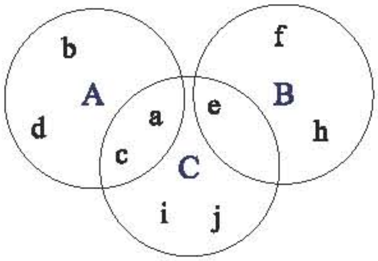
xi. If n (B) = t, then number of binary relations in $B \times B$ is:

(a) t (b) t^2 (c) 2^t (d) 2^{t^2}

2. If R = {a, f, h, s} and S = {b, e, j, n}, then

(i) Find the number of binary relations in R × S.
(ii) Write any 3 binary relations from R × S.
(iii) Write a binary relation whose domain is equal to set R.
(iv) Write a binary relation whose range is equal to set S.
(v) Write a binary relation whose domain is equal to set R and range is equal to set S.

3. Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$, $A - (B \cap C)$ and $B - (A \cup C)$, in the following Venn diagram.



4. Verify associative properties of union and intersection through Venn diagram for $X = \{2x \mid x \in N \wedge x < 20\}$, Y = Set of first 6 natural multiples of 3, $Z = \{6x \mid x \in W \wedge x < 20\}$
5. Verify distributive law of union over intersection through Venn diagram for the following sets.

   $A = \{1, 2, 3, ...\}$, $B = \{-1, -2, -3, ...\}$, $C = \{0, \pm 1, \pm 2, \pm 3, ...\}$
6. Verify distributive law of intersection over union through Venn diagram for the following sets.

   $P = \{a, b, c, d, e\}$, $Q = \{c, d, e, f\}$, $R = \{g, h, i, j, k\}$
7. Create a Venn diagram to illustrate the following information regrading the subsets A and B in the universal set.
   (i) $n(A) = 60, n(B) = 48, n(A \cap B) = 20, n(U) = 90$
   (ii) $n(A) = 34, n(B) = 52, n(A \cup B) = 60, n(U) = 85$
8. 10 boys participated in a Qiraat competition. Among them, Haani, Zubair and Haider recited in Naafi Qiraat style, Abdullah, Umer, Bilal and Ali recited in Al-Kissai Qiraat style while Hassan, Jaffer and Usman recited in Al-Kufi Qiraat style. Represent boys participated in Naafi, Al-Kissai and Al-Kufi styles by sets A, B and C respectively. Find:
   (i) Find tabular form of A, B and C.
   (ii) Draw Venn diagram of situation.
   (iii) Find $A \cap (B \cap C)^c$, $(A \cup B) \cap C^c$, $A - (B \cap C)$ and $A - (A \cup B)$.
9. 100 candidates were appeared in an examination. Out of which 45 candidates passed in Mathematics, 40 in Science and 50 in Health. If 12 were passed in Mathematics and Science, 15 in Science and Health, 20 in Health and Mathematics and 5 were passed in all three subjects.
   (i) Illustrate the above information by drawing a Venn diagram.
   (ii) How many candidates were passed at least one subject?
   (iii) How many candidates did not pass any subject?

UNIT

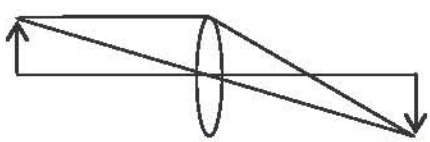
04

FACTORIZATION AND ALGEBRAIC MANIPULATION

In this unit the students will be able to:

- Recall factorization of expressions of the following types.
  $ka + kb + kc,\ ac + ad + bc + bd,\ a^2 \pm 2ab + b^2,\ \ a^2 - b^2,\ \ a^2 \pm 2ab + b^2 - c^2$
- Factorize the expressions of the following types:
  $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$, $\quad x^2 + px + q, \quad ax^2 + bx + c,$
  $(ax^2 + bx + c)(ax^2 + bx + d) + k, \quad (x + a)(x + b)(x + c)(x + d) + k$
  $(x + a)(x + b)(x + c)(x + d) + kx^2$
  $a^3 + 3a^2b + 3ab^2 + b^3, \quad a^3 - 3a^2b + 3ab^2 - b^3, \quad a^3 \pm b^3$
- Find highest common factor (HCF) and least common multiple (LCM) of algebraic expressions.
- Use factor or division method to determine highest common factor and least common multiple.
- Know the relationship between HCF and LCM.
- Solve real life problems related to HCF and LCM.
- Use highest common factor and least common multiple to reduce fractional expressions involving $+, -, \times, \div$.
- Find square root of algebraic expression by factorization and division.

Ansel Adams (1902 – 1984) was a famous American photographer known for his style of detailed and focused photos that showed its subjects simply and directly. To take sharp and clear pictures, Adams had to focus the camera precisely. The distance from the object to the lens 'p' and the distance from the lens to the film 'q' must be calculated accurately to ensure that sharp image. The focal length of the lens is 'f'. The formula that relates these measurements is $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$. This formula involves addition of two algebraic fractions with the help of LCM.



4.1 Factorization of an Algebraic Expression

The process in which an algebraic expression can be expressed as the product of its factors is called its factorization. For example,

$$15x^2y = (5x)(3xy)$$

Here, $5x$ and $3xy$ are the factors of $15x^2y$.

Hence, the factorization process which converts expressions like $15x^2y$ into $(5x)(3xy)$ is essentially the opposite of the multiplication process.

Multiply

Polynomial = Factors

$15x^2y$ $(5x)(3xy)$

Factorize

In the previous grades, we have learnt about the factorization of polynomials of the following types:

(i)	$ka + kb + kc = k(a + b + c)$
(ii)	$ac + ad + bc + bd = (c + d)(a + b)$
(iii)	$a^2 \pm 2ab + b^2 = (a \quad b)^2$
(iv)	$a^2 - b^2 = (a + b)(a - b)$
(v)	$a^2 + 2ab + b^2 - c^2 = (a + b + c)(a + b - c)$
(vi)	$a^2 - 2ab + b^2 - c^2 = (a - b + c)(a - b - c)$

Let us learn some more about the factorization of polynomials.

Type-I → **Factorizing Expressions of the Forms**

(a) $a^4 + a^2b^2 + b^4$ **(b)** $a^4 + 4b^4$

(a) To factorize $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$, we shall modify it and try to make it in appropriate form to utilize the previous results for its factorization.

Consider: $a^4 + a^2b^2 + b^4 = (a^2)^2 + a^2b^2 + (b^2)^2$

Here first and last terms are perfect squares but middle term is not twice the product of the square root of first and last term. So, we shall add and subtract a^2b^2 to make it twice.

i.e. including zero term

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^2)^2 + a^2b^2 + a^2b^2 + (b^2)^2 - a^2b^2 \\ &= (a^2)^2 + 2a^2b^2 + (b^2)^2 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \end{aligned}$$

So, $a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$

(b) Similarly, $a^4 + 64b^4 = (a^2)^2 + (8b^2)^2$

zero term

$$\begin{aligned} &= (a^2)^2 + 2(a^2)(8b^2) + (8b^2)^2 - 2(a^2)(8b^2) \\ &= (a^2 + 8b^2)^2 - 16a^2b^2 \\ &= (a^2 + 8b^2)^2 - (4ab)^2 \\ &= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab) \end{aligned}$$

Example 1: Factorize the expression $8x^4 - 26x^2m^2 + 18m^4$.

Solution:

$$8x^4 - 26x^2m^2 + 18m^4$$
$$= 2(4x^4 - 13x^2m^2 + 9m^4)$$
$$= 2[(2x^2)^2 - 13x^2m^2 + (3m^2)^2]$$
$$= 2[(2x^2)^2 + 2(2x^2)(3m^2) + (3m^2)^2 - 2(2x^2)(3m^2) - 13x^2m^2]$$
$$= 2[(2x^2 + 3m^2)^2 - 12x^2m^2 - 13x^2m^2]$$
$$= 2[(2x^2 + 3m^2)^2 - 25x^2m^2]$$
$$= 2[(2x^2 + 3m^2)^2 - (5xm)^2]$$
$$= 2(2x^2 + 3m^2 + 5xm)(2x^2 + 3m^2 - 5xm)$$

Check Point

Can you factorize?
$u^4 - u^2 + 1$

Type-II → **Factorization of a Trinomial of the Form**

(i) $x^2 + px + q$ **(ii)** $ax^2 + bx + c$

To factorize a trinomial of this form means to express the trinomial as the product of two binomials. To factorize such trinomials keep in mind the following steps:

(i) Make a list of all possible factors of 'product of extreme coefficients'.

(ii) Select a pair of factors among the list such that:

- their sum is equal to middle coefficient if extremes terms have same signs.
- their difference is equal to middle coefficient if extremes terms have opposite signs.

Example 2: Factorize the following polynomials.

(a) $y^2 - 7y + 12$ **(b)** $m^2 - 2m - 15$

Solution: (a)

$$y^2 - 7y + 12$$
$$= y^2 - 4y - 3y + 12$$
$$= y(y - 4) - 3(y - 4)$$
$$= (y - 4)(y - 3)$$

Consider only negative factors when the middle term is negative and the coefficients of last term is positive.

So, $y^2 - 7y + 12 = (y - 3)(y - 4)$

product of –3 and –4

sum of –3 and –4

(b)

$$m^2 - 2m - 15$$
$$= m^2 - 5m + 3m - 15$$
$$= m(m - 5) + 3(m - 5)$$
$$= (m - 5)(m + 3)$$

Consider one positive and one negative factor when the coefficient of middle term and of last term are negative.

Example 3: Factorize $3x^2 + 22x - 16$ ← when middle term is positive & last term is negative.

Solution:

$$3x^2 + 22x - 16$$
$$= 3x^2 + 24x - 2x - 16$$
$$= (3x^2 + 24x) - (2x + 16)$$
$$= 3x(x + 8) - 2(x + 8)$$
$$= (x + 8)(3x - 2)$$

EXERCISE 4.1

Factorize the following polynomials.

1. $2x^2y^3 - 6x^2y^2 + 2xy^3$ **2.** $3nx - 3x - 3ny + 3y$

3. $18x^4 + 108x^2y^2 + 162y^4$ **4.** $(k+2)^2 - 8(k+2) + 16$

5. $9x^2 + 4 - 169y^2 - 12x$ **6.** $(x^2-1)(y+1) - (y+3)(x^2-1)$

7. $x^2 - 6ax + 9a^2 - 16b^2$ **8.** $1 - x^2 - 2xy - y^2$

9. Find a polynomial whose factorization is $(x+y-2c)(x+2c+y)$ by using an appropriate formula.

10. Show the expression $x^2 + 4y^2 - z^2 + 4xy$ as the difference of two squares.

11. Find the missing factor in the following.

(a) $(2y^2 - 3y - 27) = (y+3)(\quad\quad)$ **(b)** $(5x^2 + 12x - 9) = (\quad\quad)(x+3)$

Factorize the following expressions.

12. $x^4 + 4m^4$ **13.** $m^4 + m^2 + 1$ **14.** $3x^4 \quad 21x^3 + 24x^2$

15. $x^8 + x^4 + 1$ **16.** $4x^4 + 256y^4$ **17.** $12 - 7x + x^2$

18. $x^2 - 9x + 8$ **19.** $10z^2 - 29z + 10$ **20.** $-3y^2 + 13y - 4$

21. $x^2 - 21x + 90$ **22.** $x^2 + x - 2$ **23.** $3x^2 + 11x + 6$

24. $2x^2 - 5xy - 3y^2$ **25.** $8 + 6x - 5x^2$ **26.** $6 - 7x - 5x^2$

27. $2a^2 - 4a - 6$ **28.** $u^4 - 13u^2 + 36$ **29.** $y^4 - 12y^2 - 64$

Type-III → **Factorizing Expressions of the Forms**

(a) $(ax^2 + bx + c)(ax^2 + bx + d) + k$

(b) $(x+a)(x+b)(x+c)(x+d) + k$

(c) $(x+a)(x+b)(x+c)(x+d) + kx^2$

The process of factorizing expressions of above types will be explained in the following examples.

(a) $(ax^2 + bx + c)(ax^2 + bx + d) + k$

Example 4: Factorize: $(x^2 + 3x - 4)(x^2 + 3x + 5) + 8$

We observe here that first two terms inside both the parentheses are same. i.e.

$$(x^2 + 3x - 4\quad (x^2 + 3x + 5)\ + 8$$

same

Let $x^2 + 3x = a$, then above expression will take the form

$$\begin{aligned}
&= (a-4)(a+5) + 8 \\
&= a(a+5) - 4(a+5) + 8 \\
&= a^2 + 5a - 4a - 20 + 8 = a^2 + a - 12 \\
&= a^2 + 4a - 3a - 12 \\
&= (a^2 + 4a) - (3a + 12) \\
&= a(a+4) - 3(a+4) = (a+4)(a-3) \\
&= (x^2 + 3x + 4)(x^2 + 3x - 3) \text{ .....by putting the value of a.}
\end{aligned}$$

(b) $(x+a)(x+b)(x+c)(x+d)+k$

To factorize such expressions, consider the the following examples.

Example 5a: Factorize: $(x+5)(x+3)(x+2)(x+6)-88$

Solution: $(x+5)(x+3)(x+2)(x+6)-88$ …… notice here, $5+3=2+6$

$$=[(x+5)(x+3)][(x+2)(x+6)]-88$$
$$=(x^2+8x+15)(x^2+8x+12)-88$$

Let $x^2+8x=a$, then above expression will take the form

$$=(a+15)(a+12)-88$$
$$=a^2+27a+180-88$$
$$=a^2+27a+92$$
$$=a^2+4a+23a+92$$
$$=(a^2+4a)+(23a+92)$$
$$=a(a+4)+23(a+4)=(a+4)(a+23)$$

$\therefore$ By back substitution

$$=(x^2+8x+4)(x^2+8x+23)$$

Example 5b: Factorize the expression

$$(z+1)(z-5)(z-9)(z-3)+44$$

Solution:

$$(z+1)(z-5)(z-9)(z-3)+44$$

Combine $(z+1)$ with $(z-9)$ and $(z-5)$ with $(z-3)$.

Re-arranging the given expression, we have

$$=(z+1)(z-9)(z-5)(z-3)+44$$
$$=(z^2-8z-9)(z^2-8z+15)+44$$

By putting $z^2-8z=x$ in the above expression,

it will take the form

$$=(x-9)(x+15)+44$$
$$=x^2-9x+15x-135+44$$
$$=x^2+6x-91$$
$$=x^2+13x-7x-91$$
$$=(x^2+13x)-(7x+91)$$
$$=x(x+13)-7(x+13)$$
$$=(x+13)(x-7)$$

Now replacing x by z^2-8z, we have

$$=(z^2-8z+13)(z^2-8z-7)$$

Check Point

While assuming same binomials equal to another variable, you must have to consider that variable which is not already present in the given expression.
Do you know why?

(c) $(x+a)(x+b)(x+c)(x+d)+kx^2$

Example 6: **Factorize the expression.**

$$(x+1)(x+2)(x+3)(x+6)-3x^2$$

Solution: $(x+1)(x+6)(x+2)(x+3)-3x^2$

$$=(x^2+6+7x)(x^2+6+5x)-3x^2$$

let $x^2+6=y$

$$=(y+7x)(y+5x)-3x^2$$
$$=y^2+5xy+7xy+35x^2-3x^2$$
$$=y^2+12xy+32x^2$$
$$=y^2+8xy+4xy+32x^2$$
$$=y(y+8x)+4x(y+8x)$$
$$=(y+8)(y+4x)$$

putting the value of y

$$=(x^2+6+8x)(x^2+6+4x)$$
$$=(x^2+8x+6)(x^2+4x+6)$$

Type-IV → **Factorizing expressions of the forms:**

(a) $a^3 + 3a^2b + 3ab^2 + b^3$
(b) $a^3 - 3a^2b + 3ab^2 - b^3$

We have studied in the previous unit that:

(i) $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$
(ii) $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$

The technique of factorization is elaborated through following examples.

Example 7: Factorize the following expressions.

(i) $27a^3 + 1 + 27a^2 + 9a$

(ii) $x^3 + \frac{3}{x} - \frac{1}{x^3} - 3x$

(iii) $4x^2(2x - 15) - 25(5 - 6x)$

Solution:

(i) $27a^3 + 1 + 27a^2 + 9a$

$= (3a)^3 + (1)^3 + 3(3a)^2(1) + 3(3a)(1)^2$
$= (3a)^3 + 3(3a)^2(1) + 3(3a)(1)^2 + (1)^3$
$= (3a + 1)^3$

(iii) $4x^2(2x - 15) - 25(5 - 6x)$

$= 8x^3 - 60x^2 - 125 + 150x$
$= 8x^3 - 60x^2 + 150x - 125$
$= (2x)^3 - 3(2x)^2(5) + 3(2x)(5)^2 - (5)^3$
$= (2x - 5)^3$

(ii) $x^3 + \frac{3}{x} - \frac{1}{x^3} - 3x$

$= x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$

$= (x)^3 - 3(x)^2\left(\frac{1}{x}\right) + 3(x)\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^3$

$= \left(x - \frac{1}{x}\right)^3$

Type-V → **Factorizing the sum and difference of two cubes**

(a) $a^3 + b^3$ (b) $a^3 - b^3$

We have studied in the previous unit that:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad \text{and} \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example 8: Factorize: $x^3 + 27y^3$

Solution:

As, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

So, $x^3 + 27y^3 = (x)^3 + (3y)^3 = (x + 3y)[(x)^2 - (x)(3y) + (3y)^2]$
$= (x + 3y)(x^2 - 3xy + 9y^2)$

Example 9: Factorize $x^6 - y^6$.

Solution: $x^6 - y^6 = (x^3)^2 - (y^3)^2$
$= (x^3 + y^3)(x^3 - y^3)$

$$= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$$
$$= (x+y)(x-y)(x^2-xy+y^2)(x^2+xy+y^2)$$

Example 10: Factorize $8a^3 - 125b^3 - 2a + 5b$

Solution: $8a^3 - 125b^3 - 2a + 5b$

$$= (2a)^3 - (5b)^3 - (2a - 5b)$$
$$= (2a - 5b)\ [(2a)^2 + (2a)(5b) + (5b)^2] - (2a - 5b)$$
$$= (2a - 5b)\ (4a^2 + 10ab + 25b^2) - (2a - 5b)$$
$$= (2a - 5b)\ (4a^2 + 10ab + 25b^2 - 1) \longleftarrow \text{factor out } (2a - 5b)$$

EXERCISE 4.2

Factorize the following expressions completely.

1. $x^3 - 125$
2. $8x^3 + 1$
3. $3p^3q^3 - 81x^3$
4. $27 + 512x^3$
5. $t^6 - 64$
6. $x^6 + y^6$
7. $(2-x)^3 + (y-2)^3$
8. $64(x+y)^3 - z^3$
9. $27p^3 + 144pq^2 - 108p^2q - 64q^3$
10. $8p^3 + q^3 + 12p^2q + 6pq^2$
11. $125x^3 - y^3 - 75x^2y + 15xy^2$
12. $p^3 - 9p^2q + 27pq^2 - 27q^3$
13. $(2x^2 - 3x + 6)(2x^2 - 3x) - 55$
14. $(y^2 + 2y - 3)(y^2 + 2y + 11) + 48$
15. $y(y-1)(y-3)(y-4) + 2$
16. $(k+2)(k-3)(k+5)(k+10) + 375$
17. $(x-5)(x-6)(x+3)(x+2) + 12$
18. $(x+1)(x+2)(x-3)(x-6) - 21x^2$
19. $(x-2)(x-6)(x-3)(x-4) - 2x^2$
20. $(5-x)(2+x)(10-x)(1+x) - 7x^2$
21. The expression $a^6 + 729$ can be written in two ways as :
   (a) sum of two squares **(b)** sum of two cubes,
   which one will be used for factoring it and why? Also factorize the given expression.
22. Express $8 + 12t + 6t^2 + t^3$ as the product of three factors. Is each factor a binomial or a trinomial?

4.2 Highest Common Factor of Algebraic Expressions

As algebra is an extension of arithmetic, so we apply almost the same rules for finding HCF of two or more algebraic expressions (polynomials) as used in arithmetic.

4.2.1 Highest Common Factor by Factorization

A factor of a polynomial is an-other polynomial which divides it completely. The common factor of two or more polynomials is a polynomial which divides them exactly.

> The highest common factor of two or more than two polynomials is a highest degree polynomial which divides the given polynomials exactly.

(a) HCF of Monomial Expressions

To Find the HCF of Monomials:
I. Determine the HCF of numerical coefficients by prime factorization.
II. Determine the common variables and select their lowest power that appears in all monomials.
This will be the HCF of variables.

Example 11: Find the HCF of $18ab^3c$, $30a^2b^4c^3$, $24b^2c^5$.
Solution: First we find the HCF of numerical coefficients 18, 30 and 24 as

$$18 = 2 \times 3^2$$
$$30 = 2 \times 3 \times 5$$
$$24 = 2^3 \times 3$$

$\therefore$ HCF of 18, 30 and 24 = $2 \times 3 = 6$
HCF of b^2, b^3 and b^4 is b^2. ⟵ least common power of b
HCF of c, c^3 and c^5 is c. ⟵ least common power of c
Hence, required HCF = $6b^2c$

Food for Thought
What is HCF of $7x^2$ and $5y^2$?

(b) HCF of Compound Polynomial Expressions

To Find the HCF of Compound Expressions:
I. Write each expression in complete factored form. Repeated factors should be expressed as powers.
II. Select the least power of each common factor.
III. The highest common factor (HCF) is the product of results of step-II.

Example 12: Find the HCF of $2m^2 - 2mn$, $4m^4 - 4m^2n^2$ and $2m^3 - 4m^2n + 2mn^2$.
Solution: First factor each polynomial expression completely as

$$2m^2 - 2mn = 2m(m - n)$$
$$4m^4 - 4m^2n^2 = 4m^2(m^2 - n^2)$$
$$= 2^2m^2(m + n)(m - n)$$
$$2m^3 - 4m^2n + 2mn^2 = 2m(m^2 - 2mn + n^2)$$
$$= 2m(m - n)^2$$

Common factors with least power are 2, m, m – n

$\therefore$ Required HCF = $2m(m - n)$

Example 13: Find the HCF of the following.

$$ax^2 + 7ax + 12a,\ ax^2 - 5ax - 24a,\ 2ax^2 + 5ax - 3a$$

Solution: $ax^2 + 7ax + 12a = a(x^2 + 7x + 12)$
$= a(x^2 + 4x + 3x + 12)$
$= a[x(x + 4) + 3(x + 4)]$
$= a(x + 3)(x + 4)$

$ax^2 - 5ax - 24a = a(x^2 - 5x - 24)$
$= a(x^2 - 8x + 3x - 24)$
$= a[x(x - 8) + 3(x - 8)]$

$$= a\,(x+3)\,(x-8)$$
$$2ax^2 + 5ax - 3a = a\,(2x^2 + 5x - 3)$$
$$= a\,(2x^2 + 6x - x - 3)$$
$$= a\,[\,2x\,(x+3) - 1\,(x+3)]$$
$$= a\,(2x-1)\,(x+3)$$
$\therefore$ Required HCF = product of all common factors with least power
$$= a\,(x+3)$$

Food for Thought

What would be the HCF of a^3-b^3 and a^3+b^3?

EXERCISE 4.3

1.(a) Find the HCF of the following monomials by completing the table.

Monomials	HCF of numerical coefficients	HCF of 'p'	HCF of 'q'	HCF of 'r'	Required HCF
$16p^3q$, $9pq^2r$					
$10p^3q^2r$, $5p^2qr$, $15p^2qr^2$					
$14p^4qr^4$, $28p^3qr^2$, $7p^2qr^2$, $21p^2q^2r^4$					

(b) If all common factors with least power of three unknown polynomials are 2^2, 3, pq and $(p+q)^2$ then what would be their HCF?

(c) Write any two polynomials of your choice having HCF as 1.

(d) The only common factor of two polynomials is $m-n$ and the only uncommon factor is m^2+n^2. Can you guess the unknown polynomials?

(e) Can you guess HCF of two polynomials x^3+5x+1 and $1+5x+x^3$ without any procedure?

Find the HCF of the following by factorization.

2. $(x+y)^2$, x^2-y^2
3. $(a-b)^3$, $a^2-2ab+b^2$
4. a^3b-ab^3, $a^5b^2-a^2b^5$
5. x^2-49, $x^2-4x-21$
6. $12x^2+x-1$, $15x^2+8x+1$
7. $c^2x^2-d^2$, $acx^2-bcx+adx-bd$
8. m^2-n^2, m^4-n^4, m^6-n^6
9. $ax^2+2a^2x+a^3$, $2ax^2-4a^2x-6a^3$, $3\,(ax+a^2)^2$

4.2.2 Highest Common Factor by Division

Sometimes, it is difficult to factorize the given polynomials completely. In such cases, we adopt division method to find the HCF of these polynomials.
We will explain the procedure with the help of the following example.

Example 14: Find the HCF of two polynomials x^3+x^2-5x+3 and x^2+3x.

Solution: The following steps will be followed for finding HCF.

Step-I: Arrange the given polynomials in descending order w.r.t. the variables.

In this case, polynomials are already in descending order.

Step-II: Consider the higher degree polynomial as dividend and lower degree polynomial as divisor and start division process.

Step-III: Since degree of remainder has become smaller than the degree of divisor, so remainder will be taken as divisor and divisor will be considered as dividend.

$$x^2+3x\,\big)\,x^3+x^2-5x+3\,\big(x-2$$
$$\underline{\pm x^3 \pm 3x^2}$$
$$-2x^2-5x+3$$
$$\underline{\mp 2x^2 \mp 6x}$$
$$x+3\,\big)\,x^2+3x\,\big(x$$
$$\underline{\pm x^2 \pm 3x}$$
$$0$$

Divisor

Continue the same process until we get 0 remainder. As $x+3$ is the last divisor which gives remainder as '0'. Therefore, required HCF $= x+3$

Example 15: Find the HCF of $12+16a+7a^2+a^3$ and $13a+5a^2+14+a^3$.

Solution: First arrange the given polynomials in descending order w.r.t. the variable. i.e. $a^3+7a^2+16a+12$ and $a^3+5a^2+13a+14$. Since degree of both the polynomials is same, so any one can be taken as a divisor or dividend.

$$a^3+5a^2+13a+14\,\big)\,a^3+7a^2+16a+12\,\big(1$$
$$\underline{\pm a^3 \pm 5a^2 \pm 13a \pm 14}$$
$$2a^2+3a-2\,\big)\,a^3+5a^2+13a+14\,\big(a+7$$
$$\underline{\times 2}$$
$$2a^3+10a^2+26a+28$$
$$\underline{\pm 2a^3 \pm 3a^2 \mp 2a}$$
$$7a^2+28a+28$$
$$\underline{\times 2}$$
$$14a^2+56a+56$$
$$\underline{\pm 14a^2 \pm 21a \mp 14}$$
$$35a+70 = 35(a+2)$$

Since first term of the divisor is $2a^2$ and first term of the dividend is a^3. Therefore, for convenience, first we multiply the dividend by 2.

Again first we will multiply the remainder by 2.

Since 35 is not a factor of any polynomial so we can neglect it for convenience.

Now divide $2a^2+3a-2$ by $a+2$.

$$a+2\,\big)\,2a^2+3a-2\,\big(2a-1$$
$$\underline{\pm 2a^2 + 4a}$$
$$-a-2$$
$$\underline{\mp a \mp 2}$$
$$0$$

$\therefore$ Required HCF $= a+2$

Key Fact

HCF of two polynomials P and Q will be the same as that of mP and nQ (where m and n are non zero constants). So in the process of finding the HCF, we can multiply or divide any divisor or dividend by any suitable number according to our requirement, but we cannot multiply or divide it by a variable.

4.3 Least Common Multiple of Algebraic Expressions

> The least common multiple of two or more polynomials is a polynomial of lowest degree that contains the factors of each polynomial.

4.3.1 Least Common Multiple by Factorization

The process of determining the LCM is almost identical to that for determining the HCF. Prime factorization is also useful to determine the LCM of two or more polynomials. The LCM is obtained by taking product of all factors (common and uncommon) with highest power.

(a) LCM of Monomial Expressions

To determine the LCM of two or more monomials, find LCM of the numerical coefficients and LCM of each variable. Then find their product for the required LCM.
To understand this concept, consider the two monomials $45x^2y$ and $60x^3y^2z$.
First find the LCM of numerical coefficients 45 and 60 which is 180.
Now consider variables x, y and z.

LCM of x^2 and x^3 = highest power of x appearing in any monomial = x^3
LCM of y and y^2 = highest power of y appearing in any monomial = y^2
LCM of z = highest power of z appearing in any monomial = z

Thus, required LCM = product of LCM of coefficients and LCM of each variable.
$= 180x^3y^2z$

Example 16: Find LCM of $18a^3b^4c^5$, $60a^3b^4c^6$ and $42a^4b^3$.

Solution: First we find the LCM of 18, 60 and 42 by prime factorization as

$$18 = 2\times 3^2,\quad 60 = 2^2\times 3\times 5,\quad 42 = 2\times 3\times 7$$

$\therefore$ LCM of 18, 60 and 42 = product of all factors with highest power
$= 2^2\times 3^2\times 5\times 7 = 1260$

Now we find LCM of each variable as

LCM of a^3 and a^4 = a^4 (highest power of a)
LCM of b^3 and b^4 = b^4 (highest power of b)
LCM of c^5 and c^6 = c^6 (highest power of c)
$\therefore$ Required LCM = $1260a^4b^4c^6$

Thinking Zone

Haleema has $60(x+1)$ english story books, $45(x+1)$ math fun books and $75(x+1)$ science fun books. She wants to put all books in groups of same number. What do you think can be the biggest number of books to be put in each group.

(b) LCM of Compound Polynomial Expressions

To understand the procedure, let us consider the following examples.

Example 17: Find LCM of $x^4y^3 - x^3y^4$ and $x^3y - xy^3$

Solution: First we factorize each polynomial completely.

$$x^4y^3 - x^3y^4 = x^3y^3(x-y)$$

$$x^3y - xy^3 = xy(x^2-y^2) = xy(x-y)(x+y)$$

We choose every factor with highest power as: x^3y^3, $x-y$ and $x+y$.

Then their product is $x^3y^3(x+y)(x-y)$

$\therefore$ Required LCM $= x^3y^3(x+y)(x-y)$
$= x^3y^3(x^2-y^2)$
$= x^5y^3 - x^3y^5$

To find the LCM of Compound Expressions:

I. Write each polynomial in complete factored form. Repeated factors should be expressed as powers.

II. Select the highest power of every factor that appears.

III. The least common multiple is the product of the results of step II.

Example 18: Find the LCM of $2x^2 - 12xy + 16y^2$, $x^2 - 6xy + 8y^2$ and $3x^2 - 12y^2$.

Solution:

$$2x^2 - 12xy + 16y^2 = 2(x^2 - 6xy + 8y^2)$$
$$= 2(x-4y)(x-2y)$$
$$x^2 - 6xy + 8y^2 = (x-4y)(x-2y)$$
$$3x^2 - 12y^2 = 3(x^2 - 4y^2)$$
$$= 3(x+2y)(x-2y)$$

All factors with highest powers are 2, 3, $x - 4y$, $x - 2y$, $x + 2y$

$\therefore$ Required LCM $= 2 \times 3 \times (x-4y)(x-2y)(x+2y)$

$= 6(x-4y)(x^2-4y^2)$

Note: Same pattern will be followed for more than three polynomials.

Key Fact

- The key words in the two processes of HCF and LCM are:
  HCF ⟶ lowest power and common factor.
  LCM ⟶ highest power and every factor.
- LCM of two or more polynomials is a lowest degree polynomial that is exactly divisible by each of the given polynomials.

EXERCISE 4.4

1. Give quick answers to these questions without doing any procedure.
   If HCF of two polynomials $x^3 + 5x^2 + 6x$ and $x^3 + 9x^2 + 14x$ is obtained as $x^2 + 2x$, then:

(i) What would be the HCF of $5(x^3 + 5x^2 + 6x)$ and $x^3 + 9x^2 + 14x$?

(ii) What would be the HCF of $x^3 + 5x^2 + 6x$ and $2(x^3 + 9x^2 + 14x)$?

(iii) What would be the HCF of $3(x^3 + 5x^2 + 6x)$ and $7(x^3 + 9x^2 + 14x)$?

(iv) What would be the HCF of $15(x^3 + 5x^2 + 6x)$ and $25(x^3 + 9x^2 + 14x)$?

(v) Does HCF of the given polynomials will remain unchanged if both are multiplied by x?

2. Find the LCM of the following monomials by completing the table.

Monomials	LCM of numerical coefficients	LCM of 'x'	LCM of 'y'	LCM of 'z'	Required LCM
(i) $8x^6y,\ 4x^2yz$					
(ii) $12x^2y^4z, 24x^3z$					
(iii) $18x^3z, 9xy^2z, 6x^6yz^3$					
(iv) $xyz^3, z^5y^3x, 28x^3y^5z$					

Find the HCF of the following by division method.

3. $a^2 + a - 2,\ a^3 + 2a^2 + a + 2$
4. $x^3 + 2x^2 - 4x - 8,\ 2x^3 + 7x^2 + 4x - 4$
5. $2x^3 + x^2 - x - 2,\ 3x^3 - x^2 + x - 3$
6. $3 + 2p^4 + 5p^2,\ 5p + 5p^3 + 3 + 3p^2$
7. $24x^4 - 2x^3 - 60x^2 - 32x,\ 18x^4 - 6x^3 - 39x^2 - 18x$
8. $2x^3 + 6x^2 + x + 3,\ 3x^3 + 9x^2 - 2x - 6,\ x^3 + 3x^2 + 2x + 6$

Find the LCM of the following expressions.

9. $9a^2b - b,\ 6a^2 + 2a$
10. $p^3q - pq^3,\ p^5q^2 - p^2q^5$
11. $4x^2y - y,\ 2x^2 + x$
12. $x^2 - x - 6,\ x^2 + x - 2,\ x^2 - 4x + 3$
13. $m^6 - 1,\ m^4 - 1,\ m^3 - 1$
14. $x^3 + 2x^2 - x - 2,\ x^2 - x - 2,\ x^2 - 4$
15. $x^2 + x - 20,\ x^2 - 10x + 24,\ x^2 - x - 30$

4.3.2 Relation between HCF and LCM

Consider two polynomials $P = a^2 - b^2$ and $Q = a^2 - 2ab + b^2$. For their LCM and HCF, first factorize them as

$$P = a^2 - b^2 = (a + b)(a - b)$$
$$Q = a^2 - 2ab + b^2 = (a - b)^2$$

$\therefore$ HCF of P and Q = $a - b$

and LCM of P and Q = $(a + b)(a - b)^2$

Now $\quad$ HCF × LCM $= (a - b)(a + b)(a - b)^2$

$= (a^2 - b^2)(a - b)^2$

$= (a^2 - b^2)(a^2 - 2ab + b^2)$ .... (i)

Also, product of P and Q $= (a^2 - b^2)(a^2 - 2ab + b^2)$ .... (ii)

Thus, $\quad$ LCM × HCF = P × Q .... (1)

Hence, it can be generalized that:

product of their HCF and LCM = product of given polynomials

4.3.3 Finding of Least Common Multiple by Division

Sometimes, it is much difficult to find the LCM of given polynomials P and Q by factorization method. Then in that case, we can find the LCM by division as follows. From the relation (1) we have,

$$\text{LCM} = \frac{P\times Q}{HCF} = \frac{P}{HCF}\times Q = \frac{Q}{HCF}\times P$$

Procedure is illustrated through examples.

Example 19: Find the LCM of P = $10x^4 + 3x^3 + 8$ and Q = $8x^4 + 3x + 10$

Solution: First we will find their HCF as

$$\begin{array}{l} \qquad\qquad\qquad\qquad 5 \\ 8x^4+3x+10\overline{)10x^4+3x^3+8} \\ \qquad\qquad\quad \times 4 \longleftarrow \text{multiplying dividend by 4} \\ \qquad\qquad\quad 40x^4+12x^3+32 \\ \qquad\qquad\quad \underline{\pm 40x^4 \quad \pm 50 \pm 15x} \\ \qquad\qquad\qquad 12x^3-18-15x \end{array}$$

where $(12x^3 - 18 - 15x) \div 3 = 4x^3 - 6 - 5x = 4x^3 - 5x - 6$

Now, we will divide $8x^4 + 3x + 10$ by $4x^3 - 5x - 6$

$$\begin{array}{l} \qquad\qquad\qquad 2x \\ 4x^3-5x-6\overline{)\ 8x^4+3x+10} \\ \qquad\qquad \underline{\pm 8x^4 \mp 12x \quad \mp 10x^2} \\ \qquad\qquad\qquad 15x+10+10x^2 = 10x^2+15x+10 \end{array}$$

Again, $(10x^2 + 15x + 10) \div 5 = 2x^2 + 3x + 2$

$$\begin{array}{l} \qquad\qquad\qquad 2x-3 \\ 2x^2+3x+2\overline{)\ 4x^3-5x-6} \\ \qquad\qquad \underline{\pm 4x^3 \pm 4x \quad \pm 6x^2} \\ \qquad\qquad -6x^2-9x-6 \\ \qquad\qquad \underline{\mp 6x^2 \mp 9x \mp 6} \\ \qquad\qquad\qquad 0 \end{array}$$

$\therefore$ HCF is $2x^2 + 3x + 2$.

Now, we obtain the LCM using the following relation.

$$\text{LCM} = \frac{P\times Q}{HCF} = \frac{(10x^4+3x^3+8)(8x^4+3x+10)}{2x^2+3x+2}$$

(Since, the HCF of two polynomials divides both of them exactly. So divide any polynomial of numerator by denominator.)

$$\begin{array}{l} \qquad\qquad\qquad 5x^2-6x+4 \\ 2x^2+3x+2\overline{)\ 10x^4+3x^3+0x^2+0x+8} \\ \qquad\qquad \underline{\pm 10x^4 \pm 15x^3 \pm 10x^2} \\ \qquad\qquad\qquad -12x^3-10x^2+0x+8 \\ \qquad\qquad\qquad \underline{\mp 12x^3 \mp 18x^2 \mp 12x} \\ \qquad\qquad\qquad\qquad 8x^2+12x+8 \\ \qquad\qquad\qquad\qquad \underline{\pm 8x^2 \pm 12x \pm 8} \\ \qquad\qquad\qquad\qquad\qquad 0 \end{array}$$

Memory Plus

HCF is not affected by multiplying or dividing any polynomial with any number during the process of finding HCF.

Hence, required LCM $= (5x^2 - 6x + 4)(8x^4 + 3x + 10)$

Example 20: Find the second polynomial Q when first polynomial $P = x^2 - 5x + 6$,
HCF $= x - 3$ and LCM $= x^3 - 9x^2 + 26x - 24$

Solution: $P = x^2 - 5x + 6$, $Q = ?$

LCM of P and Q $= x^3 - 9x^2 + 26x - 24$

HCF $= x - 3$

$$Q = \frac{HCF \times LCM}{P}$$

$$Q = \frac{(x-3)(x^3 - 9x^2 + 26x - 24)}{x^2 - 5x + 6}$$

$$Q = \frac{(x-3)(x^3 - 9x^2 + 26x - 24)}{(x-3)(x-2)}$$

$$Q = \frac{x^3 - 9x^2 + 26x - 24}{x-2} = x^2 - 7x + 12$$

$$\begin{array}{r|l} & x^2 - 7x + 12 \\ x-2 & x^3 - 9x^2 + 26x - 24 \\ & \underline{\pm x^3 \mp 2x^2} \\ & -7x^2 + 26x - 24 \\ & \underline{\mp 7x^2 \pm 14x} \\ & 12x - 24 \\ & \underline{\pm 12x \mp 24} \\ & 0 \end{array}$$

EXERCISE 4.5

1. Find the HCF and LCM of the following.
   i. $16 - 4x^2,\ x^2 + x - 6$ ii. $a^4 - a^3 - a + 1,\ a^4 + a^2 + 1$
   iii. $x^3 + 2x^2 - 3x,\ 2x^3 + 5x^2 - 3x$
2. If HCF and LCM of two polynomials are $x - 7$ and $x^3 - 10x^2 + 11x + 70$ respectively. Then find product of two polynomials.
3. Product of two polynomials is $x^4 + 3x^3 - 12x^2 - 20x + 48$ and their HCF is $x - 2$. Find their LCM.
4. The product of two polynomials is $y^4 + 6y^3 - 3y^2 - 56y - 48$ and their LCM is $y^3 + 2y^2 - 11y - 12$. Find their HCF.
5. Find the second polynomial when,
   First polynomial $= x^4 + x^3 + x + 1$, HCF $= x + 1$ and LCM $= (x^3 + 1)(x^4 + x^3 - x - 1)$
6. Find the LCM of polynomials $4x^3 - 10x^2 + 4x + 2$ and $3x^4 - 2x^3 - 3x + 2$ if their HCF is $x - 1$.

4.4 Basic Operations on Algebraic Fractions

In this topic addition, subtraction, multiplication and division of algebraic fractions will be discussed.

4.4.1 Algebraic Fractions

Algebraic expressions of the type $\dfrac{x^2 + 2x + 1}{2x^2 - 5x - 3}, \dfrac{-1 - t}{\frac{1}{t} + 1}, \dfrac{p^2}{qp^2}$

are called algebraic fractions.

4.4.2 Multiplication of Algebraic Fractions

We multiply two or more algebraic fractions in the same way as common fractions in arithmetic.

If $\frac{R}{T}$ and $\frac{S}{U}$ are two algebraic fractions, where $T \neq 0, U \neq 0$.

Then, $\frac{R}{T} \times \frac{S}{U} = \frac{RS}{TU}$

In general, before finding the product of two or more algebraic fractions, we factorize the numerator and denominator of each fraction, if possible, and divide out all the common factors, to get the most reduced form of the resulting fraction.

Key Fact

In the process of multiplication, product of all the common factors being cancelled is in fact HCF of the expressions present in numerator and denominator.

Example 21: Find the indicated product.

$$\frac{x^2-1}{x^2+4x+4} \times \frac{x+2}{x^2+2x-3}$$

Pointer to Ponder

The key to success in simplifying an algebraic fraction lies in your ability to factor the polynomials.

Solution:

$$\frac{x^2-1}{x^2+4x+4} \times \frac{x+2}{x^2+2x-3}$$

$$= \frac{(x+1)(x-1)}{(x+2)(x+2)} \times \frac{x+2}{(x-1)(x+3)}$$

$$= \frac{x+1}{x+2} \times \frac{1}{x+3}$$

$$= \frac{x+1}{(x+2)(x+3)} = \frac{x+1}{x^2+5x+6}$$

4.4.3 Division of Algebraic Fractions

If $\frac{S}{U}$ and $\frac{T}{Q}$ are two algebraic fractions, then

$$\frac{S}{U} \div \frac{T}{Q} = \frac{S}{U} \times \frac{Q}{T} = \frac{SQ}{UT}, \text{ where } U \neq 0, Q \neq 0 \text{ and } T \neq 0$$

To divide one fraction by another, invert the divisor and proceed as in multiplication because 'to divide' by a fraction means 'to multiply' by its reciprocal.

Example 22: Find the indicated operation.

$$\frac{r^2 - s^2}{r} \div \frac{r-s}{s}$$

Solution: $= \dfrac{r^2 - s^2}{r} \div \dfrac{r-s}{s}$

$$= \frac{(r+s)(r-s)}{r} \times \frac{s}{r-s} \quad = \quad \frac{r+s}{r} \times \frac{s}{1}$$

$$= \frac{s(r+s)}{r} \quad = \quad \frac{sr+s^2}{r}$$

Example 23: Simplify: $\dfrac{3x^3+6x^2}{2x^2+x-6} \div (6x^2-15x)$

Solution: $\dfrac{3x^3+6x^2}{2x^2+x-6} \div (6x^2-15x)$

$$= \frac{3x^3+6x^2}{2x^2+x-6} \times \frac{1}{6x^2-15x}$$

$$= \frac{3x^2(x+2)}{(x+2)(2x-3)} \times \frac{1}{3x(2x-5)}$$

$$= \frac{x}{(2x-3)(2x-5)} \quad = \quad \frac{x}{4x^2-16x+15}$$

4.4.4 Addition and Subtraction of Algebraic Fractions

Algebraic fractions are added or subtracted just like arithmetic fractions i.e., by manipulating LCM of their denominators.

Example 24: Simplify the following.

$$\frac{x^2+2x+2}{2x} + \frac{(-2x-2)}{2x}$$

Solution: $\dfrac{x^2+2x+2}{2x} + \dfrac{(-2x-2)}{2x}$

$$= \frac{x^2+2x+2+(-2x-2)}{2x}$$

$$= \frac{x^2+2x+2-2x-2}{2x}$$

$$= \frac{x^2}{2x} \quad = \quad \frac{x}{2}$$

Enlighten Yourself

Algebraic fractions with same denominators are called like fractions e.g., $\dfrac{a-2}{a+b}, \dfrac{b^2-2}{a+b}$

Fractions with different denominators are called unlike fractions e.g.,

$$\frac{x^2-y}{ax+b}, \frac{y^2-x}{cx+d}$$

Example 25: Perform the indicated operations.

$$\frac{-2x}{x+3}+\frac{3}{3-x}-\frac{8x-12}{x^2-9}$$

Solution: $\frac{-2x}{x+3}+\frac{3}{3-x}-\frac{8x-12}{x^2-9}$

$$=\frac{-2x}{x+3}+\frac{-3}{x-3}-\frac{8x-12}{(x+3)(x-3)}$$ ⟵ $\because 3-x=-(x-3)$

$$=\frac{-2x(x-3)+(-3)(x+3)-(8x-12)}{(x+3)(x-3)}$$ ⟵ $\because$ LCM is $(x+3)(x-3)$

$$=\frac{-2x^2+6x-3x-9-8x+12}{(x+3)(x-3)}$$ ⟵ simplify the parentheses

$$=\frac{-2x^2-5x+3}{(x+3)(x-3)}$$ ⟵ combine like terms

$$=\frac{-1(2x^2+5x-3)}{(x+3)(x-3)}$$ ⟵ negative sign is taken as common

$$=\frac{-1(2x-1)(x+3)}{(x+3)(x-3)}$$

$$=\frac{-(2x-1)}{(x-3)}$$

$$=-\frac{2x-1}{x-3}$$ ⟵ reduced form

4.4.5 Algebraic Fractions with Combined Operations

When two or more operations occur in any algebraic expression then the rule for order of operations (DMAS) must be followed.

Example 26: Simplify.

$$\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2u+6v}{u^2-4v^2}\div\frac{u^2+6uv+9v^2}{u^2v-2uv^2}$$

Solution: Division is performed before addition while simplifying an expression. Therefore, first we will simplify fractions having '÷' sign.

$$\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2u+6v}{u^2-4v^2}\div\frac{u^2+6uv+9v^2}{u^2v-2uv^2}$$

$$=\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2u+6v}{u^2-4v^2}\times\frac{u^2v-2uv^2}{u^2+6uv+9v^2}$$

$$=\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2(u+3v)}{(u+2v)(u-2v)}\times\frac{uv(u-2v)}{(u+3v)^2}$$

$$=\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2uv}{(u+2v)(u+3v)}$$

$$= \frac{3u^2+5uv+2v^2}{(u+2v)(u+3v)} + \frac{2uv}{(u+2v)(u+3v)}$$

$$= \frac{3u^2+5uv+2v^2+2uv}{(u+2v)(u+3v)}$$

$$= \frac{3u^2+7uv+2v^2}{(u+2v)(u+3v)}$$

$$= \frac{(3u+v)(u+2v)}{(u+2v)(u+3v)} = \frac{3u+v}{u+3v}$$

EXERCISE 4.6

1. Answer these without calculations.

i. Product of what algebraic fraction and x^3+7x-8 is 1?

ii. Which algebraic fraction divided by $\frac{x^2}{x^2+y^2}$ gives 1?

iii. Sum of what algebraic fraction and $\frac{m}{m^2+n^2}$ is $\frac{m+n}{m^2+n^2}$?

iv. What is the product of an algebraic fraction and its reciprocal?

Simplify the following (where all the expressions in the denominator are non-zero).

2. $\frac{14x^2-7x}{12x^3+24x^2} \times \frac{x^2+2x}{2x-1}$

3. $\frac{a^2b^2+3ab}{4a^2-1} \times \frac{2a+1}{ab+3}$

4. $\frac{a-b}{a^2+ab} \times \frac{a^4-b^4}{a^2-2ab+b^2} \times \frac{a}{a^2+b^2}$

5. $\frac{6x^2y^2}{x^2-y^2} \div \frac{3xy}{x+y}$

6. $\frac{8a^3-1}{4a^3+2a^2} \div \frac{6a^2-13a+5}{15a-25} \times \frac{2a^4+a^3}{15a^2}$

7. $\frac{x^2-8x-9}{x^2-17x+72} \times \frac{x^2-25}{x^2-1} \div \frac{x^2+4x-5}{x^2-9x+8}$

8. $\frac{1}{2x-3y} - \frac{x+y}{4x^2-9y^2}$

9. $\frac{1}{x(x-y)} + \frac{1}{y(x+y)}$

10. $\frac{5x+5}{3(2x-1)} + \frac{6-2x}{2(1-2x)}$

11. $\frac{2a}{2a-3} - \frac{5}{6a+9} - \frac{4(3a+2)}{3(4a^2-9)}$

12. $\frac{5}{5+x-18x^2} - \frac{2}{2+5x+2x^2}$

13. $\frac{1-p^2}{1+q} \times \frac{1-q^2}{p+p^2} \times \left(1+\frac{p}{1-p}\right)$

4.5 Square Root of Algebraic Expressions

The square root of an algebraic expression is defined as one of its equal factors e.g., $x + y$ is the square root of $x^2 + 2xy + y^2$ because,

$$x^2 + 2xy + y^2 = (x + y)^2$$

> The square root of an algebraic expression P is another algebraic expression Q which, when squared, gives P. Thus, if $P = (-Q)^2$ then Q and – Q both are square roots of P.

Square root of an algebraic expression can be obtained in two ways.

i. Factorization Method   ii. Division Method

4.5.1 Square Root by Factorization Method

In this method, before applying the square root, given expression is written in the form of a complete square. For example, to find the square root of $4a^2 + 12ab + 9b^2$, first we will convert the given expression into a complete square as follows:

$$\begin{aligned} 4a^2 + 12ab + 9b^2 &= (2a)^2 + 2(2a)(3b) + (3b)^2 \\ &= (2a + 3b)^2 \\ &= [\pm\ (2a + 3b)]^2 \end{aligned}$$

Now, applying square root on both sides, we get.

$$\sqrt{4a^2 + 12ab + 9b^2} = \pm (2a + 3b)$$

Key Fact

The square root of an algebraic expression consists of two expressions, which are additive inverses of each other.

Example 27: Find the square root of $\left(a + \frac{1}{a}\right)^2 - 4\left(a - \frac{1}{a}\right)$.

Solution: To find square root of such type of expressions, we can adopt two methods.

Method-I

$$\begin{aligned} \left(a + \frac{1}{a}\right)^2 - 4\left(a - \frac{1}{a}\right) &= a^2 + 2(a)\left(\frac{1}{a}\right) + \frac{1}{a^2} - 4\left(a - \frac{1}{a}\right) \\ &= a^2 + 2 + \frac{1}{a^2} - 4\left(a - \frac{1}{a}\right) \\ &= a^2 - 2 + \frac{1}{a^2} - 4\left(a - \frac{1}{a}\right) + 2 + 2 \\ &= \left(a - \frac{1}{a}\right)^2 - 4\left(a - \frac{1}{a}\right) + 4 \\ &= \left(a - \frac{1}{a}\right)^2 - 2\left(a - \frac{1}{a}\right)(2) + (2)^2 \\ &= \left(a - \frac{1}{a} - 2\right)^2 \end{aligned}$$

$$\sqrt{\left(a + \frac{1}{a}\right)^2 - 4\left(a - \frac{1}{a}\right)} = \pm\left(a - \frac{1}{a} - 2\right) \text{....(applying square root)}$$

Method-II

$$\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right) = a^2+2+\frac{1}{a^2}-4\left(a-\frac{1}{a}\right)$$

$$= \left(a^2+\frac{1}{a^2}\right)+2-4\left(a-\frac{1}{a}\right) \quad \text{........ (i)}$$

Let, $a-\frac{1}{a}=x$ .... (a)

Then, $\left(a-\frac{1}{a}\right)^2=x^2$

or $a^2-2+\frac{1}{a^2}=x^2$

or $a^2+\frac{1}{a^2}=x^2+2$ .... (b)

Substituting values from equations (a) and (b) in equation (i)

$$\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right) = (x^2+2)+2-4x$$

$$= x^2-4x+4$$

$$= (x-2)^2 \quad \text{........ (ii)}$$

Now, replace x by $a-\frac{1}{a}$ in equation (ii)

$$\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)=\left(a-\frac{1}{a}-2\right)^2$$

Applying square root on both sides, we get

$$\sqrt{\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)} = \pm\left(a-\frac{1}{a}-2\right).$$

Example 28: Find the square root of $(2a+1)(2a+3)(2a+5)(2a+7)+16$.

Solution:

$$(2a+1)(2a+3)(2a+5)(2a+7)+16=[(2a+1)(2a+7)][(2a+3)(2a+5)]+16$$

$$=(4a^2+16a+7)(4a^2+16a+15)+16$$

$$=(4a^2+16a+7)(4a^2+16a+7+8)+16$$

$(2a+1)(2a+3)(2a+5)(2a+7)+16=x(x+8)+16$ .....(put $4a^2+16a+7=x$)

$$=x^2+8x+16$$

$$=x^2+2(x)(4)+(4)^2 = (x+4)^2$$

Now replace x by $4a^2+16a+7$

$$(2a+1)(2a+3)(2a+5)(2a+7)+16 = (4a^2+16a+7+4)^2 = (4a^2+16a+11)^2$$

Applying square root on both sides, we get

$$\sqrt{(2a+1)(2a+3)(2a+5)(2a+7)+16} = \pm(4a^2+16a+11)$$

4.5.2 Square Root by Division Method

To understand this method, consider the following example.

Example 29: Find the square root of $9x^2 - 42xy + 49y^2$ by division method.

Solution:

$$3x - 7y \leftarrow \text{root}$$

$\left[\sqrt{9x^2} = 3x\right] \rightarrow$	$3x$	$9x^2 - 42xy + 49y^2$
	$\underline{+3x}$	$\underline{\pm 9x^2}$
$\left[\dfrac{-42xy}{6x} = -7y\right]$	$6x - 7y$	$-42xy + 49y^2 \leftarrow$ remainder
		$\underline{\mp 42xy \pm 49y^2}$
		$0 \leftarrow$ remainder

$\therefore$ Square root $= \pm(3x - 7y)$

Example 30: Find the square root of $4\left(x^2 + \dfrac{1}{x^2}\right) + 12\left(x + \dfrac{1}{x}\right) + 17, x \neq 0.$

Solution:

$$4\left(x^2 + \frac{1}{x^2}\right) + 12\left(x + \frac{1}{x}\right) + 17 = 4x^2 + \frac{4}{x^2} + 12x + \frac{12}{x} + 17$$

$$= 4x^2 + 12x + 17 + \frac{12}{x} + \frac{4}{x^2} \leftarrow \text{arrange in descending order}$$

$$2x + 3 + \frac{2}{x}$$

$2x$	$4x^2 + 12x + 17 + \dfrac{12}{x} + \dfrac{4}{x^2}$
	$\underline{\pm 4x^2}$
$4x + 3$	$+12x + 17 + \dfrac{12}{x} + \dfrac{4}{x^2}$
	$\underline{\pm 12x \pm 9}$
$4x + 6 + \dfrac{2}{x}$	$8 + \dfrac{12}{x} + \dfrac{4}{x^2}$
	$\underline{\pm 8 \pm \dfrac{12}{x} \pm \dfrac{4}{x^2}}$
	0

$\therefore$ Required square root is $\pm\left(2x + 3 + \dfrac{2}{x}\right)$

Sometimes the given expression is not a perfect square and to make it a perfect square, we add (subtract) some value to (from) it. Consider the following example.

Example 31: To convert $x^4 + 4x^3 + 10x^2 + 10x + 5$ into a perfect square,

i. What should be added to it?
ii. What should be subtracted from it?
iii. What should be the value of x?

Solution: First we try to find the square root of the given expression as

$$
\begin{array}{r|l}
 & \quad x^2 + 2x + 3 \\
\hline
x^2 & x^4 + 4x^3 + 10x^2 + 10x + 5 \\
 & \underline{\pm x^4} \\
2x^2 + 2x & \quad 4x^3 + 10x^2 + 10x + 5 \\
 & \quad \underline{\pm 4x^3 \pm 4x^2} \\
2x^2 + 4x + 3 & \qquad 6x^2 + 10x + 5 \\
 & \qquad \underline{\pm 6x^2 \pm 12x \pm 9} \\
 & \qquad\qquad -2x - 4
\end{array}
$$

For a perfect square, remainder must be zero, but here remainder is $-2x-4$.
Hence,

i. Expression will be a perfect square if we add $-(-2x-4) = 2x+4$ in it.
ii. Expression will be a perfect square if we subtract $-2x-4$ from it.
iii. For the value of x put remainder equal to zero as

$$-2x - 4 = 0$$
$$-2x = 4 \quad \text{or} \quad x = -2$$

$\therefore$ For $x = -2$, expression will be a perfect square.

Example 32: For what values of m and n, $9x^4 - 24x^3 - 14x^2 + mx + n$ is a complete square or perfect squre?

Solution: First we try to find the square root of $9x^4 - 24x^3 - 14x^2 + mx + n$.

$$
\begin{array}{r|l}
 & \quad 3x^2 - 4x - 5 \\
\hline
3x^2 & 9x^4 - 24x^3 - 14x^2 + mx + n \\
 & \pm 9x^4 \\
\hline
6x^2 - 4x & -24x^3 - 14x^2 + mx + n \\
 & \mp 24x^3 \pm 16x^2 \\
\hline
6x^2 - 8x - 5 & -30x^2 + mx + n \\
 & \mp 30x^2 \pm 40x \pm 25 \\
\hline
 & mx - 40x + n - 25 \\
 & = x(m - 40) + (n - 25)
\end{array}
$$

The, given expression will be a complete square if the remainder is zero. This is only possible if,

$m - 40 = 0$ and $n - 25 = 0$

or $m = 40$ and $n = 25$

Hence, for $m = 40$ and $n = 25$ given expression will be a complete square

EXERCISE 4.7

Find the square root of the following by factorization.

1. $16y^2 - 56y + 49$

2. $25a^4 - 30a^3 + 9a^2$

3. $\left(x^2 - \dfrac{1}{x^2}\right)^2 + 4\left(x^2 - \dfrac{1}{x^2}\right) + 4, \ x \neq 0$

4. $\left(a^2 + \dfrac{1}{a^2}\right) - 8\left(a - \dfrac{1}{a}\right) + 14, \ a \neq 0$

5. $(a + 2)(a + 4)(a + 6)(a + 8) + 16$

Find the square root of the following by division.

6. $x^4 + 8x^3 + 20x^2 + 16x + 4$

7. $x^4 + 10x^3 + 31x^2 + 30x + 9$

8. $49b^4 + 18a^2b^2 + 4a^3b + a^4 + 28ab^3$

9. $4x^4 - 12x^3 + 29x^2 - 30x + 25$

10. $1 - 10x + 27x^2 - 10x^3 + x^4$

11. $x^4 + \dfrac{1}{x^4} + 4x^2 - \dfrac{4}{x^2} + 2, \ x \neq 0$

12. $a^2 - 8a + 2 + \dfrac{56}{a} + \dfrac{49}{a^2}, \ a \neq 0$

13. $x^4 - 2x^3 + \dfrac{3x^2}{2} - \dfrac{x}{2} + \dfrac{1}{16}$

14. $4x^4 + 32x^2 + 96 + \dfrac{64}{x^4} + \dfrac{128}{x^2}$

15. To make $a^4 - 10a^3 + 27a^2 - 9a + 2$ a perfect square:
   i. What should be added in it? ii. What should be subtracted from it?
   iii. What will be the value of a?
16. Find the values of p and q if $x^4 - 12x^3 + px + q$ is a complete square.
17. For what value of k, the expression $y^4 + 4y^2 + k + \frac{8}{y^2} + \frac{4}{y^4}$ becomes a perfect square, where $y \neq 0$.

4.5.3 Application of Factorization in Daily Life

We use various basic principles of mathematics quite unknowingly in our daily life. Like, we are always using addition, subtraction, division and multiplication everywhere, from restaurants to public transport. When children learn about numbers and basic mathematics. This happens because, with time, we become familiar with the concepts. They can frequently apply these concepts in solving their real life problems. Factorization is a similar example of this, we have a bunch of real-life examples where we use factorization extensively, making our daily lives easier.

Example 33:
Mustafa is working on a space project to make a cuboid with the volume as the difference of 2 cubes of sides x and y respectively. Help him to find:

a. Expression for volume of the required cuboid.
b. Expression for any one side of required cuboid.
c. Expression for Area of any one surface of required cuboid.

Solution:
Mustafa is working with cubes of volumes:
Volume 1: x^3 Volume 2 : y^3
Difference of 2 volumes = $x^3 - y^3$

a. Volume of required cuboid = $x^3 - y^3$
b. $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ (by factorizing)
   The volume of a cuboid is factorized in one linear and one quadratic factor. So the expression for length is "$x-y$"
c. The quadratic factor in above factorization represents Area of a surface. i-e Area of one surface is $x^2 + xy + y^2$.

Example 34:
Volume of a cubical container is given by expression $(x^3 - 6x^2y + 12xy^2 - 8y^3)\ m^3$. Find:

a. Expression for length of each side
b. Expression for area of the base
c. Expression for total surface area
d. Expression for cost of painting all outer surfaces at the rate of Rs.50/m^2.

Solution:

Volume of cube = $l^3 = x^3 - 6x^2y + 12xy^2 - 8y^3$

$= (x - 2y)^3$

a. Length of each side is $(x - 2y)$ m

b. Base area = $l^2 = (x - 2y)^2\ m^2$

c. Surface area = $6l^2 = 6\ (x - 2y)^2\ m^2$

d. Cost = rate × surface area = Rs. $50 \times 6 \times (x - 2y)^2$

= Rs. $300\ (x - 2y)^2$

EXERCISE 4.8

1. In a map of an industry, expression of area of a rectangular veranda is given by $(x^2 - 2x - 3)m^2$. Find:
   a. expressions for both dimensions of veranda.
   b. expression for perimeter of veranda.
   c. expression for cost of fencing veranda @ Rs. 200/m.
   d. expression for the cost of carpeting veranda floor @ Rs. 250/m².

2. Area of a square shaped surface of a machine is given by the expression $(25x^2 - 30x + 9)m^2$. Find:
   a. expression for the length of the surface.
   b. expression for the boundary of the surface.
   c. expression for the cost of polishing the surface of machine @ Rs75/m².
   d. expression for the cost of edging around 2 sides of the machine surface @ Rs 28/m.

3. Volume of a cubical oil tank in an oil refinery is expressed as $(125x^3 - 150x^2 + 60x - 8)\ m^3$. Find:
   a. expression for height of oil tank.
   b. expression for surface area of oil tank.
   c. expression for painting it from outside @ Rs32/m².

4. A mechanical engineer working on wheels of a machine, finds their areas given by

   $A_1 = \pi x^2 - 6\pi x + 9\pi$ and $A_2 = \pi x^2 - 10\pi x + 25\pi$. Help him find radii of both wheels.

5. A machine has two squared shape pressers with areas expressed by $25m^2$ and $36n^2$ respectively. Difference of these areas describes a rectangular presser. Find dimension of the rectangular pressers.

6. Distance covered by a missile to hit the target is given by expression $(x^2 + 5x + 6)$m. Find:
   a. the possible expression for speed of missile
   b. the possible expression for time to reach the target

KEY POINTS

- Factorization of expressions of the following types.

Type-I: $ka + kb + kc = k(a + b + c)$

Type-II: $ac + ad + bc + bd = (a + b)(c + d)$

Type-III: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$

Type-IV: $a^2 - b^2 = (a + b)(a - b)$

Type-V: $a^2 + 2ab + b^2 - c^2 = (a + b + c)(a + b - c)$ and

$a^2 - 2ab + b^2 - c^2 = (a - b + c)(a - b - c)$

Type-VI: $a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$

$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$

Type-VII: $x^2 + px + q$ (factorize it write p as the sum of the factors of q)

Type-VIII: $ax^2 + bx + c$ (factorize it write b as the sum of the factors of ac)

Type-IX:
$(ax^2 + bx + c)(ax^2 + bx + d) + k$
$(x + a)(x + b)(x + c)(x + d) + k$ (where $a + b = c + d$)
$(x + a)(x + b)(x + c)(x + d) + kx^2$

Type-X: $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$ and $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$

Type-XI: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

- HCF of two or more polynomials is a highest degree polynomial which divides the given polynomials exactly.
- LCM of two or more polynomials is a least degree polynomial which is exactly divisible by the given polynomials.
- Product of two polynomials P and Q = Product of their HCF and LCM i.e.
  $P \times Q = HCF \times LCM$
- An algebraic expression of the form $\frac{P}{Q}$ where P and Q are two expressions and $Q \neq 0$ is called an algebraic fraction.
- A fraction having rational expression in its denominator or numerator or both, is called a complex fraction.
- In case of division, after converting the fractions into multiplication form, follow the same rules for
  simplification as applied in the product of algebraic fractions.
- Use 'DMAS' rule while simplifying the algebraic fractions having more than one operation.
- Square root of an algebraic expression is defined as one of its equal factors.
- Square root of an algebraic expression 'P' is another expression 'Q' which, when squared, gives 'P'
  i.e., if $P = (\pm Q)^2$ then, $+Q$ and $-Q$ both are square roots of P.

MISCELLANEOUS EXERCISE 4

1. Encircle the correct option in the following.

(i) Factors of $-2-a+a^2$ are

(a) $(a-2)(a-1)$ (b) $(a+1)(a+2)$ (c) $(a+2)(a-1)$ (d) $(a+1)(a-2)$

(ii) Factorization of $x^2-x+\frac{1}{4}$ is

(a) $\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$ (b) $\left(x+\frac{1}{2}\right)(x-1)$

(c) $\left(x-\frac{1}{2}\right)(x+1)$ (d) $\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$

(iii). $(x^2+m^2)(x+m)(x^4+m^4)(x-m)$ is the factored form of

(a) x^4-m^4 (b) x^8+m^8 (c) x^8-m^8 (d) x^4+m^4

(iv). a^4+64b^4 is the product of

(a) $a^2-4ab+8b^2$ and $a^2+4ab+8b^2$ (b) a^2+8b^2+4ab and a^2-8b^2+4ab

(c) $(a^2+8b^2)^2$ (d) none of these

(v). $x(2a+3b+7)+x(2a+3b+5)$ equals

(a) $2x(2a+3b+6)$ (b) $4x(2a+3b+6)$

(c) $x(2a+3b+7)(2a+3b+5)$ (d) $(x+x)(2a+3b+12)$

(vi) Which is the highest common factor of $-12x^2y^2$, $6xy^3$, $24x^2y^2$?

(a) $-6xy$ (b) $6x^2y^3$ (c) $6xy^2$ (d) $-24x^2y^3$

(vii) What is the least common multiple of $12x^2y^2$, $6xy^3$, $24x^2y^2$?

(a) $6xy$ (b) $6x^2y^3$ (c) $6xy^2$ (d) $24y^3x^2$

(viii) What is the highest common factor of $7x-6xy$ and $5xy^3-3x^2$?

(a) $(7-6y)(5y^3-3x)$ (b) $(7x-6xy)(5y^3x-3x^2)$

(c) x (d) $x(7-6y)(5y^3-3x)$

(ix) Least common multiple of $7x-6xy$ and $5y^3x-3x^2$ is:

(a) $(7-6y)(5y^3-3x^2)$ (b) $(7x-6xy)(5y^3-3x^2)$

(c) x (d) $x(7-6y)(5y^3-3x)$

(x) HCF of $7x^3-8y^3$ and $3x^3-5y^3$ is:

(a) 1 (b) $(7x^3-8y^3)(3x^3-5y^3)$

(c) $7x^3-8y^3$ (d) $3x^3-5y^3$

(xi) LCM of $7x^3 - 8y^3$ and $3x^3 - 5y^3$ is:

(a) 1 (b) $(7x^3 - 8y^3)(3x^3 - 5y^3)$

(c) $7x^3 - 8y^3$ (d) $3x^3 - 5y^3$

(xii) What is the product of $\frac{uv^2}{3w^3}$ and $\frac{6w^4}{u^2v^3}$?

(a) $\frac{uv^2}{3w^2} \times \frac{u^2v^3}{6w^4}$ (b) $\frac{2w}{uv}$ (c) $\frac{uv}{2w}$ (d) both a & c

(xiii) What is the quotient of $\frac{3y^2}{10} \div \frac{y^3}{2}$?

(a) $\frac{3y^5}{20}$ (b) $\frac{3}{5y}$ (c) $\frac{3y^2}{10} \div \frac{2}{y^3}$ (d) $\frac{5y}{3}$

(xiv) What is the sum of $\frac{2a}{a^2-1}$ and $\frac{-a}{a^2-1}$?

(a) $\frac{3a}{a^2-1}$ (b) $\frac{a}{a^2-1}$ (c) $\frac{2a-a}{(a^2-1)+(a^2-1)}$ (d) $\frac{-2a}{a^2-1}$

(xv) What is the difference of $\frac{-3x}{x+y}$ and $\frac{x}{x+y}$?

(a) $\frac{-2x}{x+y}$ (b) $\frac{-3x-x}{2x+2y}$ (c) $\frac{3x^2}{x+y}$ (d) $\frac{-4x}{x+y}$

(xvi) If product of two polynomials is $(a - b)^2 (a^2 + ab + b^2)$ and their HCF is $a - b$, what is their LCM?

(a) $a^3 - b^3$ (b) $(a - b)^3(a^2 + ab + b^2)$

(c) $(a - b)^2(a^2 + ab + b^2)$ (d) $a^2 + ab + b^2$

(xvii) If product of HCF and LCM of two polynomials is $(x^3 - y^3)(x + y)$ then what will be the product of these polynomials?

(a) $x^4 - y^4$ (b) $(x - y)(x^2 + y^2)$

(c) $(x^2 - y^2)(x^2 + xy + y^2)$ (d) $(x^2 - y^2)(x^2 - xy + y^2)$

(xviii) What is the square root of $36x^6y^{16}$?

(a) $6x^6y^{16}$ (b) $6xy$ (c) $6x^3y^8$ (d) $16x^3y^8$

(xix) What is the square root of $(15x^2 - 7y^2)^4$?

(a) $\pm(15x^2 - 7y^2)^2$ (b) $\pm(15x^2 - 7y^2)$

(c) $\pm(15x - 7y)^2$ (d) $\pm(15x - 7y)$

(xx) What is the square root of $\left[-\left(2x+\frac{1}{x}+1\right)\right]^2$?

(a) $\pm\left(2x+\frac{1}{x}+1\right)^2$ (b) $\left(2x+\frac{1}{x}+1\right)^2$ (c) $2x+\frac{1}{x}+1$ (d) $\sqrt{2x+\frac{1}{x}+1}$

Factorize the following.

2. $(a^2-5)^2-13(a^2-5)+36$
3. $6x^2+19x+15$
4. $(x+1)(x-3)(x-5)(x-9)+44$
5. x^4-12x^2+16
6. $(3x^2+4x-5)(3x^2-2+4x)-4$
7. $2m^4+m^2n^2-3n^4$
8. $x^4+10x^2y^2-56y^4$
9. $x^2+2ax-bx-2ab$
10. $a^{12}-b^{12}$
11. $28x^4y+64x^3y-60x^2y$
12. $4(x-y)^3-(x-y)$
13. $x^3p^2-8y^3p^2-4x^3q^2+32y^3q^2$
14. $x^4+y^4-7x^2y^2$
15. Find the highest common factor of the following.
    (i) $x^3+3x^2-8x-24$, x^3+3x^2-3x-9
    (ii) $3x^4-3x^3-2x^2-x-1$, $9x^4-3x^3-x-1$
16. Find the LCM of the following.
    (i) $2x^2+3x+1$, $2x^2+5x+2$, x^2+3x+2
    (ii) $3x^2+11x+6$, $3x^2+8x+4$, x^2+5x+6
17. Find the HCF and LCM of the following expressions.
    $a(a+c)-b(b+c)$, $b(b+a)-c(c+a)$, $c(c+b)-a(a+b)$
18. Find the square root of $\frac{9a^2}{x^2}-\frac{6a}{5x}+\frac{101}{25}-\frac{4x}{15a}+\frac{4x^2}{9a^2}$
19. Simplify the following (all the expressions in the denominator are non-zero).
    (i) $\frac{x^2+x-2}{x^2-x-20}\times\frac{x^2+5x+4}{x^2-x}\div\left(\frac{x^2+3x+2}{x^2-2x-15}\times\frac{x+3}{x^2}\right)$
    (ii) $\frac{1}{x+1}-\frac{1}{(x+1)(x+2)}+\frac{1}{(x+1)(x+2)(x+3)}$
20. Find the values of a and b if
    $x^4+ax^3+bx^2-4x+4$ is a perfect square.

UNIT
05

LINEAR EQUATIONS AND INEQUALITIES

In this unit the students will be able to:

- Recall linear equation in one variable.
- Solve linear equation with rational coefficients.
- Reduce equations, involving radicals, to simple linear form and find their solutions.
- Define absolute value.
- Solve the equation, involving absolute value, in one variable.
- Define inequalities (>, <) and (≥, ≤).
- Recognize properties of inequalities (i.e. trichotomy, transitive, additive and multiplicative).
- Solve linear inequalities with rational coefficients.

We can use equations and formulae to model a variety of real life problems and situations. Business, industry, science, sports, travel, architecture and banking are some of the areas that really depend upon equations and inequalities to find the solutions to their problems. For example, consider the following problem taken from our daily life. Sarim and Raahim are football players in their school team.

If we want to compare their scores for the season, only one of the following statements will be true.

- Sarim scored less umber of goals than Raahim.
- Sarim scored the same number of goals as Raahim.
- Sarim scored more number of goals than Raahim.

Let x and y represent the number of goals scored by Sarim and Raahim respectively. We can compare their scores by using an inequality or an equation as given below.

$x < y$ or $x = y$ or $x > y$.

5.1 Linear Equations in One Variable

A **linear equation in one variable** is an equation that can be written in the standard form as **ax + b = 0** where a, b are real numbers and $a \neq 0$ e.g.
$2x+3=0$, $5x+3=5$

Key Fact

1. Exponent of the variable in linear equation is always 1. The equation $x^2=4$ is non-linear because exponent of the variable is not '1'.
2. Linear equations do not involve product of the variables. The equation $xy=14$ is a non-linear equation.
3. Linear equation is also called **equation of degree one**.

5.1.1 To Find the Solution of Linear Equations

The **solution** of a linear equation in one variable is a replacement for the variable that makes the equation true.

A linear equation in one variable (in standard form) has exactly one solution. For the solution of such equations, we have to isolate the variable on either side of equal sign by a sequence of **equivalent equations.**

Key Fact

Two or more equations, which have the same solutions, are called equivalent equations. e.g. $2x+3=4$ and $2x=1$ are two **equivalent equations**.

We follow properties of equality i.e. addition, subtraction, multiplication and division properties while solving first degree linear equations.

Example 1: Solve the following equation for x.

$$-7x+24=3$$

Solution:

$-7x+24=3$ ⟵ original equation

$-7x+24-24=3-24$ ⟵ subtract 24 from both sides

$-7x=-21$ ⟵ divide both sides by -7

$x=3$

Check: To check this root, we replace the variable x by its value in the original equation and simplify both sides.

$-7x+24=3$ ⟵ original equation

$-7(3)+24\stackrel{?}{=}3$ ⟵ replace 'x' by 3

$-21+24\stackrel{?}{=}3$ ⟵ simplify L.H.S

$3=3$ ⟵ solution is checked

Thus, $x=3$ is required root.

Point to Ponder!

While checking any solution, it is better to write a question mark over the equal sign just to indicate that we are not sure of the validity of the equation.

5.1.2 Solving Linear Equations Involving Fractions

The following examples illustrate this method.

Example 2: Solve $\dfrac{3x}{5} - \dfrac{1}{2} = \dfrac{x}{4} + 1$

Solution: $\dfrac{3x}{5} - \dfrac{1}{2} = \dfrac{x}{4} + 1$

LCM of 5, 2 and 4 = 20.

$$20 \times \frac{3x}{5} - 20 \times \frac{1}{2} = 20 \times \frac{x}{4} + 20$$
$$12x - 10 = 5x + 20$$
$$12x - 5x = 20 + 10$$
$$7x = 30$$
$$x = \frac{30}{7}$$

Thus, $x = \dfrac{30}{7}$ is the required root of the given equation.

Example 3: Solve $\dfrac{2x+3}{5} = \dfrac{3-4x}{8}$

Solution: $\dfrac{2x+3}{5} = \dfrac{3-4x}{8}$

By cross multiplication we get:

$$8(2x+3) = 5(3-4x)$$
$$16x + 24 = 15 - 20x$$
$$16x + 20x = 15 - 24$$
$$36x = -9$$
$$x = -\frac{9}{36} = -\frac{1}{4}$$

Thus, $x = -\dfrac{1}{4}$ is the root of given equation.

Example 4: Solve $0.7(x-1) - 0.5x = 1.1$

Solution:

$$0.7(x-1) - 0.5x = 1.1$$
$$0.7x - 0.7 - 0.5x = 1.1$$
$$0.2x - 0.7 = 1.1$$
$$0.2x = 1.8$$
$$2x = 18$$
$$x = 9$$

Check: Replace x by its value in the original equation.

$$0.7(9-1) - 0.5(9) \stackrel{?}{=} 1.1$$
$$5.6 - 4.5 \stackrel{?}{=} 1.1$$
$$1.1 = 1.1$$

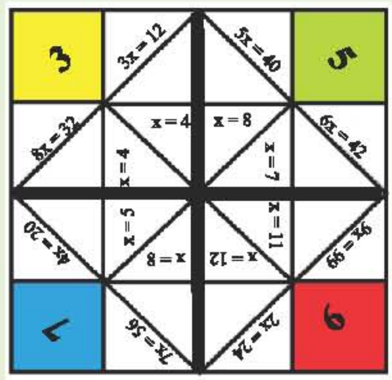
Thus, $x = 9$ is the required solution of the given equation.

Historya Mystery

Finding solution of equations has been a principal aim of mathematics for thousands of years. However, the equal sign did not occur in any text until 1557.

Funmatics

Help students to make such a cootie catcher to play with friends for reinforcement of solution of linear equations.



EXERCISE 5.1

Solve the following linear equations in one variable.

1. $5x - 2 - x = 4 - 3x - 27$
2. $4a - 3(5a - 14) = 5(7 + a) - 9$
3. $7(2 - 5x) + 27 = 18x - 3(8 - 4x)$
4. $\frac{5x}{4} + \frac{1}{2} = 0$
5. $\frac{x-2}{2} + \frac{x+10}{9} = 5$
6. $\frac{4(x+2)}{3} - \frac{6(x-7)}{7} = 12$
7. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} = 7\frac{5}{6}$
8. $\frac{y+1}{3} + \frac{y+1}{2} = 2 - \frac{y+3}{2}$
9. $\frac{1}{5}(x-8) + \frac{4+x}{7} = 7 - \frac{23-x}{5}$
10. $\frac{1}{2y} - \frac{1}{6} = \frac{1}{4y} - 1 - \frac{1}{y}$; $y \neq 0$
11. $4 - 0.3(1 - x) = 7$
12. $0.5x = 6.3 - 0.2x$
13. $1.3x - 0.2 = 0.3x - 1.5$

5.2 Linear Equations Involving Radicals

5.2.1 Radical Equations

An equation in which the unknown letter (variable) appears under a radical sign is called a **radical equation.**

Examples: $\sqrt{x+1} = 7$, $\sqrt{x} = 9$, $\sqrt{x+2} = 5$, $\sqrt{2x-3} = \sqrt{x+5}$

Key Fact

- If two numbers are equal, then their squares are also equal i.e. if $x = y$ then $x^2 = y^2$
- If the squares of two numbers are equal, the numbers may or may not be equal. e.g. $(-5)^2 = (5)^2$ but $-5 \neq 5$

5.2.2 To Solve a Radical Equation

I. Arrange the terms such that a term with a radical sign is by itself on one side of the equation.

II. Square both sides of the equation.

III. Solve the resulting linear equation for corresponding variable.

IV. Check the solution in the original equation.

Example 5: Solve $\sqrt{x}+3=7$

Solution:
$$\sqrt{x}+3=7$$
$$\sqrt{x}=4$$
$$(\sqrt{x})^2=(4)^2$$
$$x=16$$

Check: Replace x by 16 in the original equation.
$$\sqrt{16}+3\overset{?}{=}7$$
$$4+3=7$$
$$7=7 \longleftarrow \text{solution is checked}$$

Hence, $x=16$ is required solution.

> **Key Fact**
> The equation in which after isolating the radical term, if radical term is equal to a negative number, such equation has no solution in real numbers.

While squaring both sides of a radical equation it is possible to get an extra root called extraneous root that does not satisfy the original equation. Therefore, it is necessary to check every root by substituting it into the original equation. Consider the following example for such case.

Example 6: Solve $4+2\sqrt{3y+1}=3$

Solution:
$$4+2\sqrt{3y+1}=3$$
$$2\sqrt{3y+1}=-1 \quad \text{.......(i)}$$
$$(2\sqrt{3y+1})^2=(-1)^2$$
$$4(3y+1)=1$$
$$12y+4=1$$
$$12y=-3$$
$$y=-\frac{1}{4}$$

Check:
$$4+2\sqrt{3(-\frac{1}{4})+1}\overset{?}{=}3$$
$$4+2\sqrt{\frac{-3}{4}+1}\overset{?}{=}3$$
$$4+2\sqrt{\frac{1}{4}}\overset{?}{=}3$$
$$4+1\overset{?}{=}3$$
$$5\neq 3$$

Thus, $y=-\frac{1}{4}$ is an extraneous root and solution set is ϕ.

Note: In example 6, there is no need to solve the equation after step (i). We can directly say that equation has no solution.

EXERCISE 5.2

Reduce the following radical equations into simple linear equations then find their solution. In case of extraneous solution, write ϕ for the solution set.

1. $\sqrt{2x}=4$
2. $\sqrt{x-3}=2$
3. $\sqrt{x-5}=3$
4. $\sqrt{2x+1}=9$
5. $\sqrt{5x-4}=14$
6. $\sqrt{3x-5}=-10$
7. $\sqrt{y+4}-3=2$
8. $5-\sqrt{2x-1}=0$

9. $\sqrt{y+1} - 12 = -10$

10. $\sqrt{5t-2} = \sqrt{3t+4}$

11. $\sqrt{9-2x} = \sqrt{5x-12}$

12. $12 - \sqrt{y+1} = 14$

13. $4\sqrt{z} + 8 = 40$

14. $\sqrt{\dfrac{a+6}{a+2}} = \sqrt{\dfrac{a+2}{a-1}}$

15. $\sqrt{\dfrac{z}{z+3}} = \sqrt{\dfrac{z+2}{z+6}}$

16. $\sqrt{5x-4} = \sqrt{7x+2}$

5.3 Equations Involving Absolute Value

5.3.1 Defining an Absolute Value

The **absolute value** of a number is its distance from 0 on the number line. If x is any point on the number line then its distance from 0 is denoted by $|x|$. The two vertical bars are called **absolute value bars.** Since distance between any two points is always a positive number or zero, thus the absolute value of a number is always a positive number or zero e.g. distance from 0 to 5 or from 0 to – 5 is 5 units on the number line.

Thus, $|5| = 5$ and $|-5| = 5$

The absolute value of a real number x, written as $|x|$, is defined as

- $|x| = x$, if $x \geq 0$
- $|x| = -x$ if $x < 0$ e.g. $|9| = 9$ or $|-3| = -(-3) = 3$

5.3.2 Solution of Absolute Value Equations

An equation that contains a variable inside the absolute value bars is called an **absolute value equation.**

e.g. $|x| + 1 = 5$, $|x-3| = 4$

To solve the equations involving absolute value, we apply the basic definition of absolute value.

Example 7: Solve the equation $|x| = 8$.

Solution: To solve such equation we have to consider both the possible values of the number with absolute value.

Thus, if $|x| = 8$ then $x = 8$ or $x = -8$.

The solution set is $\{-8, 8\}$.

Example 8: Solve the equation $|x| = -6$.

Solution There is no real number x such that $|x| = -6$. So, this equation has no solution. Hence the solution set is ϕ.

Conclusions

An absolute value equation of the form $|ax+b| = c$, where a, b and c are real numbers, $a \neq 0$ and $c > 0$ is equivalent to two equations:

$$ax+b=c \quad \text{or} \quad ax+b=-c.$$

For the required solution set of the given absolute value equation, we solve both the equations separately.

Example 9: Solve the equation:

$$|2x+5| = 11, \text{ where } x \in R.$$

Solution:

Step-I: Remove the absolute value bars and write two equations as

$$2x+5=11 \quad \text{or} \quad 2x+5=-11$$

Step-II: Now solve both the equations for x.

$2x+5 = 11$	$2x+5 = -11$
$2x = 11-5$	$2x = -11-5$
$2x = 6$	$2x = -16$
$x = 3$	$x = -8$

Step-III: Hence 3 and -8 are roots of the absolute value equations. Thus, the solution set is $\{3, -8\}$

- The equation $|x| = k$ and $k > 0$ has two solutions k and $-k$.
- The equation $|x| = 0$ has one solution, namely $x = 0$.
- The equation $|x| = k$ and $k < 0$ has no solution and the solution set in this case is ϕ.

Example 10: Solve the absolute value equation.

$$\frac{|10-x|}{5} = \frac{|2x-5|}{2}, \quad \text{where } x \in R$$

Solution: Before we remove the absolute value bars, try to isolate the absolute value expressions on either side as

$$\frac{|10-x|}{5} = \frac{|2x-5|}{2}$$

$$\frac{|10-x|}{|2x-5|} = \frac{5}{2}$$

$$\left|\frac{10-x}{2x-5}\right| = \frac{5}{2} \longleftarrow \because \frac{|a|}{|b|} = \left|\frac{a}{b}\right|$$

$\frac{10-x}{2x-5} = \frac{5}{2}$	or $\frac{-10x}{2x-5} = -\frac{5}{2}$
$\frac{10-x}{2x-5} = \frac{5}{2}$	$\frac{10-x}{2x-5} = -\frac{5}{2}$
$2(10-x) = 5(2x-5)$	$2(10-x) = -5(2x-5)$

Key Fact

- $|ab| = |a|\,|b|$
- $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$
- $|a| + |a| = 2\,|a|$

$$20 - 2x = 10x - 25 \qquad\qquad 20 - 2x = -10x + 25$$
$$-12x = -45 \qquad\qquad 8x = 5$$
$$x = \frac{45}{12} = \frac{15}{4} \qquad\qquad x = \frac{5}{8}$$

Thus, the solution set is $\left\{\frac{15}{4}, \frac{5}{8}\right\}$.

Example 11: Solve the following absolute value equaiton.

$$|a - 1| = |2a - 3|, \quad a \in R$$

Solution:

By removing the absolute value bars we get two equations as:

$$a - 1 = 2a - 3 \quad \text{or} \quad a - 1 = -(2a - 3)$$
$$-a = -2 \qquad\qquad a - 1 = -2a + 3$$
$$a = 2 \qquad\qquad 3a = 4$$
$$a = \frac{4}{3}$$

The solution set is $\left\{2, \frac{4}{3}\right\}$.

EXERCISE 5.3

Solve the following absolute value equations, where $x, y, z \in R$.

1. $|x| = \frac{5}{3}$
2. $|x + 2| = 6$
3. $|5y - 1| = 9$
4. $|x + 1| = 2$
5. $|6 - 3y| = 0$
6. $3|z - 2| - 4 = -2$
7. $|2x - 1| = 5$
8. $|3x + 2| = 7$
9. $\frac{|4x|}{3} = 12$
10. $|5x| + 10 = 5$
11. $\frac{|1 - 2y|}{4} = 3$
12. $\frac{|x + 1|}{2} = \frac{|2x - 1|}{3}$
13. $|5x - 3| = |x + 7|$
14. $|z + 3| - 3 = 5 - |z + 3|$

5.4 Linear Inequalities (or Inequations) in one Variable

5.4.1 Defining an Inequality

There are many ways in which two expressions may be unequal. The following symbols express some inequalities.

Inequality Symbols

$<$	is less than	e.g. $-5 < 7$
$>$	is greater than	e.g. $10 > -3$
$\leq$	is less than or equal to	e.g. $7 \leq 7$
$\geq$	is greater than or equal to	e.g. $5 \geq 1$
$\neq$	is not equal to	e.g. $3 \neq 5$

History a Mystery

The symbols for "is less than" and "is greater than" were introduced by Thomas Harriot around 1630. Before that ⊏ and ⊐ were used for $<$ and $>$ respectively.

A statement that "two algebraic expressions are not equal" is called an "**inequality**" or "**inequation.**"

A **linear inequality in one variable** is an inequality (inequation) that can be written in the standard form of $ax + b < 0$ (or $ax + b > 0$) where a and b are real numbers and $a \neq 0$.

Examples: $x \leq 3$, $x \geq -2$, $x - 5 < -10$, $5y - 7 < 3y + 9$, $-5 > -7$, $-3 \neq x + 1$

Remark: Above mentioned definition is also valid for the symbols $\leq$ and $\geq$.
Some examples of linear inequations are $4x + 3 \geq 0$, $y > -7$, $8(x - 2) \leq 3 - 5x$ and $x + 3 < -5$

An inequality written with the symbols $<$ or $>$ is called a **strict inequality.**

5.4.2 Solution of Linear Inequalities (in one variable)

The definitions of **solution** and **solution set** for inequalities are same as for equations.

A **solution** of an inequality is a replacement for the variable that makes the inequality true.
Solution set of an inequality is the set of all real numbers that satisfy the inequality.

Procedure:
The procedure for solving a linear inequality in one variable is almost identical to that for solving a linear equation. Here, also to isolate the variable, we use "**properties of inequalities**". These properties are similar to the properties of equality but there is one important difference that **when both sides of an inequality are multiplied or divided by a negative number then direction of the inequality symbol is reversed.** e.g.

original inequality ⟵ $-2 < 5$

$(-3)(-2) > (-3)(5)$ ⟵ multiplying both sides by -3 and reversing inequality symbol

$6 > -15$ ⟵ simplest form

Two or more inequalities that have the same solution set are called **equivalent inequalities**
e.g. $x + 5 < 8$ and $x < 3$ are two equivalent inequalities.

Example 12: Solve the following inequality.
$8x + 9 < 6x - 7$, where $x \in$ R.

Solution:
$$8x + 9 < 6x - 7$$
$$8x - 6x < -7 - 9$$
$$2x < -16$$
$$x < -8$$

Check Point

In example 13 if $x \in$ Z, then what will be the solution set?

The solution contais all real numbers less than – 8.

Check the solution by replacing x with any number less than – 8, for example – 9, as

$$8(-9)+9<6(-9)-7$$
$$-72+9<-54-7$$
$$-63<-61 \leftarrow \text{true statement}$$

Thus the solution set is $\{x \mid x \in R \wedge x<-8\}$.

Example 13: Solve the inequality.

$$3(-4+5y) \leq -8(1-2y)+6, x \in R$$

Solution:
$$3(-4+5y) \leq -8(1-2y)+6$$
$$-12+15y \leq -8+16y+6$$
$$-12+15y \leq 16y-2$$
$$-y \leq 10$$
$$y \geq -10$$

$\therefore$ Solution Set = $\{x \mid x \in R \wedge y \geq -10\}$

Point to Ponder!

- $5 \geq 1$ is true because $5>1$ is true.
- $7 \leq 7$ is true because $7=7$ is true.

Example 14: Solve. $\frac{1}{2}x+3 \geq \frac{1}{4}x+2, x \in R$

Solution:
$$\frac{1}{2}x+3 \geq \frac{1}{4}x+2$$
$$4(\frac{1}{2}x+3) \geq 4(\frac{1}{4}x+2) \leftarrow \text{multiply both sides by LCM of denominators}$$
$$2x+12 \geq x+8$$
$$x+12 \geq 8$$
$$x \geq -4$$

The solution set is $\{x \mid x \in R \wedge x \geq -4\}$.

Compound Inequalities

Two inequalities that are joined by the word **"and"** or the word **"or"** are called compound inequalities e.g. $2x<6$ and $3x+2>-4$, $3x+5>7$ or $4x-1<3$

5.4.3 Solution of Compound Inequalities Joined with 'or'

When two inequalities are joined with connective word **'or'**, it is necessary to solve each inequality separately. The solution set of the compound inequality will be the union of both the solution sets i.e., the solution set will satisfy either one or both the inequalities.

Key Fact

A compound inequality containing or is true if at least one of its inequality is true.

Example 15: Solve the following.

$$2x+3>7 \quad \text{or} \quad 4x-1<3; \quad x \in R$$

Solution: We will solve both the inequalities for x separately.

$$2x+3>7 \quad \text{or} \quad 4x-1<3$$
$$2x>4 \quad \text{or} \quad 4x<4$$
$$x>2 \quad \text{or} \quad x<1$$

$\{x \mid x \in R \wedge x > 2\}$ or $\{x \mid x \in R \wedge x < 1\}$

Now the union of both the solution sets is

$\{x \mid x \in R \wedge x > 2 \text{ or } x < 1\}$

Hence, the solution set of the given compound inequality contains all real numbers greater than 2 or less than 1.

5.4.4 Solution of Compound Inequalities Joined with 'and'

When two inequalities are joined with the connective word **'and'** then the solution set of the compound inequality will be the intersection of both the solution sets i.e. the solution set contains all the solutions that satisfy both of the inequalities.

Example 16: Solve the compound inequality.

$x - 5 \geq -1$ and $x + 3 \leq 10, x \in R$

Solution: Solve both the inequalities for 'x'.

$x - 5 \geq -1$ and $x + 3 \leq 10$

$x \geq 4$ and $x \leq 7$

$\{x \mid x \in R \wedge x \geq 4\}$ and $\{x \mid x \in R \wedge x \leq 7\}$

Intersection of two solution sets is $\{x \mid x \in R \wedge 4 \leq x \leq 7\}$

i.e. Solution set consists of all real numbers that are greater than or equal to 4 and less than or equal to 7.

Key Fact

- Two inequalities $-6 < 5x + 3$ and $5x + 3 < 5$ can be written in combined form as $-6 < 5x + 3 < 5$.
- There is no short way to write a compound inequality containing 'or'.
- Compound inequality containing **and** is true if its both inequalities are true.

Example 17: The sum of two times a number x and 3 is between 5 and 17. Between what two numbers is the given number x?

Solution: Expression 'sum of 2 times a number x and 3' can be written as $2x + 3$. Then given compound inequality is '$5 < 2x + 3 < 17$'.

Now we solve this compound inequality.

$5 < 2x + 3 < 17$

$5 - 3 < 2x < 17 - 3$ ⟵ subtracting 3 from each part

$2 < 2x < 14$

$1 < x < 7$ ⟵ dividing each part by 2

Thus, x lies in between 1 and 7.

Math Play Ground

a. Take students to the play ground.
b. Give each student a paper strip with some equation or inequality written on.
c. Spread solutions of all students in the play ground.
d. Ask them to find the respective answers as a game of treasure hunt.

EXERCISE 5.4

1. (a) Check whether the given value of each variable satisfies the inequality.

i) $5y - 12 > 0; \quad y = 3$ ii) $4 - 2x \leq 9; \quad x = -3$

iii) $5 - 2x > -4x + 5; \quad x = 4$ v) $3(z + 4) \leq 6; \quad z = -2$

v) $5(x - 2) \geq 9x - 3(2x - 4); \quad x = 11$

(b) In the following cases, write each solution in the set notation form.

i) $2 \leq x \leq 5$, where $x \in N$ ii) $y < 7$, where $y \in N$

iii) $z \leq -3$, where $z \in R$ iv) $x \leq 4$, where $x \in W$

v) $-4 < x < \dfrac{-3}{2}$, where $x \in R$

Solve the following inequalities.

2. $3x - 2 < 7, \; x \in N$ 3. $6x - 5 \leq 35 - 2x, \; x \in W$

4. $16 - 5y < 4(y - 1) - 7, \; y \in R$ 5. $10 - (7 - y) \geq 3y - 9, \; y \in R$

Solve the following compound inequalities, where $x \in R$ (6-10).

6. $5 - 3x < 11$ or $2x + 3 < -9$ 7. $2x + 3 \leq 9$ and $x - 5 > -6$

8. $1 \leq 7 - 3x \leq 22$ 9. $3x + 21 < 1 - x$ or $3x + 8 > 3 - 2x$

10. $1 - 5x > 16$ and $3 - \dfrac{3x}{2} \leq 9$

11. The sum of five times a number x and 10 is less than –35 or greater than –5. What real numbers does x represent?

12. Two times a number decreased by 5 is greater than or equal to the number increased by 8. Find the possible values for the number.

KEY POINTS

- An equation that can be written in the standard form $ax + b = 0$ where $a, b \in R$ and $a \neq 0$ is called a linear equation in one variable.
- A solution of a linear equation in one variable is a number replacement for the variable that makes it true.
- Two or more linear equations with the same solutions are called equivalent equations.
- To solve a linear equation with fractions, multiply all the terms by the least common multiple of all denominators to clear the fractions.
- Root of a linear equation that does not satisfy the original equation is called an "extraneous root" of that equation.
- A linear equation in which the variable appears under the radical sign is called a radical equation.
- An equation that contains an absolute value symbol is called an absolute value equation.

- If x is a real number then $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.
- If two equations have the same absolute value, then they are either equal or opposite.
- A statement that two algebraic expressions are not equal is called an inequality.
- An equality that can be written in the standard form of $ax + b < 0$ where a and b are real numbers and $a \neq 0$ is called a linear inequality.
- A solution set is the set of all solutions of an inequality.
- Two or more inequalities, which have the same solution sets are called equivalent inequalities
- A compound inequality is a relation containing two simple inequalities connected with the words **'and'** or **'or'**.

MISCELLANEOUS EXERCISE 5

1. Encircle the correct option in the following. absolutely correct.

(i). Which one is the standard form of linear equation?
(a) $ax^2 + b = 0$ (b) $ax + b = -c$ (c) $ax + b > 0$ (d) $ax + b = 0$

(ii). The exponent of the variable in linear equation is?
(a) 1 (b) 2 (c) –1 (d) 0

(iii). Which one is the linear equation in one variable?
(a) $ax + y = 0$ (b) $xy + 3 = 0$ (c) $2x + 3 = 0$ (d) $2x^2 + 3 = 0$

(iv). Which one is the solution of $12x + 17 = 65$?
(a) 48 (b) $\frac{82}{12}$ (c) 4 (d) $\frac{65}{12}$

(v). What number must be subtracted from the right side of $7x = 30$ so that 4 is a solution of the resulting equation?
(a) 7 (b) 2 (c) 4 (d) –2

(vi). Which property of equality will be applied to solve the equation $-2x = \frac{2}{5}$?
(a) Addition (b) Subtraction
(c) Division (d) Addition and subtraction

(vii). Which one is the solution set of $5|x| = 25$?
(a) {–25, 25} (b) {–25} (c) {5} (d) {–5,5}

(viii). Which is the solution set of $|x| + 7 = 3$?
(a) {– 4} (b) {4, – 4} (c) { } (d) {–7, –3}

(ix). Which one is the solution set of $\sqrt{5x} = -10$?
(a) { } (b) {–20} (c) {20} (d) {–2}

(x). Which one is the solution set of $\sqrt{3x+1} = 5$?
(a) 25 (b) 8 (c) 24 (d) $\frac{26}{3}$

(xi). Which one is the solution of $\frac{4}{x}-\frac{2}{x}=5$?

(a) -1 (b) $\frac{2}{5}$ (c) $\frac{5}{2}$ (d) zero

(xii). Which one is a strict inequality?

(a) $x+3 \neq 0$ (b) $12x > 5$ (c) $2y-3 \leq 0$ (d) $4x+5 \geq 0$

(xiii). What should be value of 'k' if $x<y$ shows $kx>ky$?

(a) $k=0$ (b) $k>0$ (c) $k<0$ (d) $k \geq 0$

(xiv). Which one is the compound relation?

(a) $1+2x<4+x$ (b) $4x+3>5\frac{3}{5}$ (c) $x+y>5\frac{1}{2}$ (d) $x \leq 0$

(xv). Which one is the solution of $3-\frac{1}{2}x \geq 0$?

(a) $x \geq -6$ (b) $x \leq 6$ (c) $x \geq 6$ (d) $x \leq -6$

(xvi). Which one is the solution of $2(x+6) \leq 3(x+4)$?

(a) $x \leq 0$ (b) $x \geq 0$ (c) $x \leq 24$ (d) $x \geq 24$

Solve the following.

2. $\frac{2x-11}{12}=\frac{2x+10}{12}-\left(\frac{28-2x}{4}-\frac{1}{4}\right)$

3. $\sqrt{a-\frac{1}{2}}=\sqrt{\frac{2a}{5}+\frac{2}{5}}$

4. $\frac{\sqrt{5x-4}-4}{10}=-1$

5. $5-|5y+1|=-9$

6. $\frac{3}{4}x-1 \geq x+1, x \in R$

7. $4(2y+3)-(6y-1)>10,\ y \in R$

8. $\frac{3}{2}x \leq -3$ or $\frac{2}{3}x \geq 4,\ x \in R$

9. The difference between three times a number y and 18 is less than 12 or greater than 39. What real numbers do y represent?

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